Superfluid Density – Ward Identity Approach

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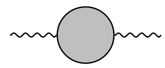
In the presence of an external gauge field, the superfluid density is given as the photon mass term generated by integrating out the fermions, and is determined by the vacuum polarization diagram

$$\frac{n_s}{m} = \quad (1)$$

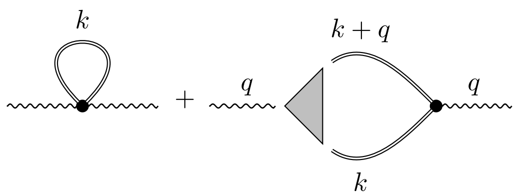


Through minimal coupling, the kinetic term of the prescribed action can be expanded as

$$\overline{\psi}\xi_{\mathbf{k}}\psi = \overline{\psi}\left(\frac{k^2}{2m} - \mu\right)\psi + \overline{\psi}\left(\frac{\mathbf{k}}{m}\cdot\mathbf{A}\right)\psi + \overline{\psi}\left(\frac{1}{2m}A^2\right)\psi, \quad (2)$$

allowing me to denote two relevant vertices; a 3-point current vertex $(-\mathbf{k}/m)$ and a 4-point vertex $(-1/m)$. For each fermion line I denote $-G(k)$ as the fully renormalized propagator. To 1-loop order,

$$\frac{n_s}{m} =$$



$$\xrightarrow{q\to 0} (-1)^3\left(\frac{1}{m}\right)\int_k G(k) + (-1)\int_k \frac{\mathbf{k}}{m}\cdot\mathbf{\Gamma}(k,k)G(k)^2 \quad (3)$$

$$= \frac{n}{m} - \int_k \frac{\mathbf{k}}{m}\cdot\mathbf{\Gamma}(k,k)G(k)^2,$$

where I neglected the dependence of fermions on momentum q. When doing this limit, it is important to let the frequency $\omega \to 0$ first, then take $\mathbf{q} \to 0$. I have taken $\Gamma(k,k)$ to be renormalized current vertex (bare value $\mathbf{k}/m$), and noted the additional minus sign for fermion bubbles. Also, $-\int G = n$ is the total fermion density.

Ward Identity - Normal Metal

In this section I will show that the Ward identity ensures the vanishing of the superfluid density (3) for the case of a normal metal [2]. This identity is very commonly derived in most QFT texts, where for a normal metal there is no anomalous term (with Δ pairing). It has the form

$$p\cdot\Gamma(k,k+p)G(k)G(k+p) = G(k+p) - G(k), \quad (4)$$

which I will prove below.

Proof: Take the general imaginary time path integral

$$\mathcal{Z} = \int \mathcal{D}[\psi,\overline{\psi}]\exp\left[-\int d^dx\,\overline{\psi}(\partial_\tau - \frac{\nabla^2}{2m} - \mu)\psi\right], \quad (5)$$

without any interaction term. From this I can write the following 2-point function

$$\langle T\psi(x_1)\overline{\psi}(x_2)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[\psi,\overline{\psi}]\psi(x_1)\overline{\psi}(x_2)\exp\left[-\int d^dx\,\overline{\psi}(\partial_\tau - \frac{\nabla^2}{2m} - \mu)\psi\right]. \quad (6)$$

I will now take the local transformation

$$\psi \to \psi e^{i\alpha(x)} \sim \psi(1+i\alpha(x)), \qquad \overline{\psi} \to \overline{\psi}e^{-i\alpha(x)} \sim \overline{\psi}(1-i\alpha(x)), \quad (7)$$

and note that the variation of the action is

$$\delta S = \int d^d x \; \partial_\mu \alpha j^\mu, \qquad\qquad j^\mu = \left(i\overline{\psi}\psi, -i(\overline{\psi}\partial_{\mathbf{x}}\psi - \psi\partial_{\mathbf{x}}\overline{\psi})/2m\right). \tag{8}$$

where $\partial_\mu = (\partial_\tau, \partial_{\mathbf{x}})$. These transformations alter (6) to

$$\begin{aligned}\left\langle T\psi(x_1)\overline{\psi}(x_2)\right\rangle &= \frac{1}{\mathcal{Z}}\int \mathcal{D}[\psi,\overline{\psi}]\psi(x_1)\overline{\psi}(x_2)\exp[-S](1+i\alpha(x_1))(1+i\alpha(x_2))\left(1-\int d^d x\; \partial_\mu \alpha j^\mu\right) \\ &= \frac{1}{\mathcal{Z}}\int \mathcal{D}[\psi,\overline{\psi}]\psi(x_1)\overline{\psi}(x_2)\exp[-S]\left(1+i\alpha(x_1)-i\alpha(x_2)-\int d^d x\; \partial_\mu \alpha j^\mu\right)\end{aligned} \tag{9}$$

where I collected terms to linear order in $\alpha(x)$. This 2-point must still be equivalent to (6), and by equating (9) with (6) I find the condition

$$\begin{aligned}0 &= \frac{1}{\mathcal{Z}}\int \mathcal{D}[\psi,\overline{\psi}]\psi(x_1)\overline{\psi}(x_2)\left(i\alpha(x_1)-i\alpha(x_2)-\int d^d x\; \partial_\mu \alpha j^\mu\right)\exp[-S] \\ &= \frac{1}{\mathcal{Z}}\int \mathcal{D}[\psi,\overline{\psi}]\psi(x_1)\overline{\psi}(x_2)\int d^d x\; i\alpha(x)\left(\delta(x-x_1)-\delta(x-x_2)-i\partial_\mu j^\mu(x)\right)\exp[-S] \\ &= \left\langle T\psi(x_1)\overline{\psi}(x_2)\int d^d x\; i\alpha(x)\left(\delta(x-x_1)-\delta(x-x_2)-i\partial_\mu j^\mu(x)\right)\right\rangle,\end{aligned} \tag{10}$$

which can be written as

$$\left\langle T\psi(x_1)\overline{\psi}(x_2) i\partial_\mu j^\mu(x)\right\rangle = \delta(x-x_1)\left\langle T\psi(x_1)\overline{\psi}(x_2)\right\rangle - \delta(x-x_2)\left\langle T\psi(x_1)\overline{\psi}(x_2)\right\rangle. \tag{11}$$

I will act with the fourier operator $\int_{x,x_1,x_2} e^{ipx}e^{ikx_1}e^{-i(k+p)x_2}$ on both sides of this relation,

$$\begin{aligned}\int_{x,x_1,x_2} e^{ipx}e^{ikx_1}e^{-i(k+p)x_2}\left\langle T\psi(x_1)\overline{\psi}(x_2)i\partial_\mu j^\mu(x)\right\rangle &= \left\langle T\psi(k+p)\overline{\psi}(k+p)\right\rangle - \left\langle T\psi(k)\overline{\psi}(k)\right\rangle \\ p_\mu G(k)\Gamma^\mu(k,k+p)G(k+p) &= G(k+p)-G(k)\end{aligned} \tag{12}$$

where I did IBP for the derivative term and noted the definition of the vertex function [3]. This is easy to see diagrammatically. Note also that $G(k) = -\left\langle \psi_k \overline{\psi}_k\right\rangle$ is the fully dressed propagator. This completes the proof.

For use in the computation of the superfluid density (3) I will first take $\omega \to 0$ in the current momentum p, then take the limit $\mathbf{p} = 0$ for (12),

$$\begin{aligned}\mathbf{p}\cdot\boldsymbol{\Gamma}(k,k+p) = -\left(G^{-1}(k+p)-G^{-1}(k)\right) \xrightarrow{\mathbf{p}\to 0} \boldsymbol{\Gamma}(k,k) &= -\partial_{\mathbf{k}}G^{-1}(k) \\ &= G^{-2}(k)\partial_{\mathbf{k}}G(k).\end{aligned} \tag{13}$$

This renormalized current vertex can be substituted into (3) to give

$$n_s = n - \int_k \mathbf{k}\cdot\partial_{\mathbf{k}}G(k) = n + \int_k G(k) = n - n = 0 \tag{14}$$

which shows that the superfluid density vanishes for a normal metal, as it should.

Ward Identity - Charge 2e Superconductor

In a BCS superconductor, the action relative to that of a normal metal has the anomalous term

$$S \supset -\int d^d x \ \left[\Delta\overline{\psi}_\uparrow(x)\overline{\psi}_\downarrow(x) + \overline{\Delta}\psi_\downarrow(x)\psi_\uparrow(x)\right] \tag{15}$$

which results in a modified Ward identity due to the breaking of U(1). The pairing with opposite spin is indicative of a spin singlet superconductor. Taking the same local transformation (7): $\psi_{\uparrow\downarrow} \to \psi_{\uparrow\downarrow}e^{i\alpha(x)}$, $\overline{\psi}_{\uparrow\downarrow} \to \overline{\psi}_{\uparrow\downarrow}e^{-i\alpha(x)}$, it is easy to show that the equivalent 2-point function (9) is

$$\begin{aligned}\left\langle T\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\right\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[\psi,\overline{\psi}]\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\exp[-S]\Big(1 + i\alpha(x_1) - i\alpha(x_2) - \int d^d x\ \partial_\mu \alpha j^\mu \\ + \int d^d x\ i\alpha(x)\left[-2\Delta\overline{\psi}_\uparrow(x)\overline{\psi}_\downarrow(x) + 2\overline{\Delta}\psi_\downarrow(x)\psi_\uparrow(x)\right]\Big).\end{aligned} \tag{16}$$

I chose $\left\langle\psi_\uparrow\overline{\psi}_\uparrow\right\rangle$ for brevity, so when computing the superfluid density n_s I will have to double it to get the full density. Seeking equivalence to the usual 2-point function, I find the condition

$$\begin{aligned}0 &= \frac{1}{\mathcal{Z}}\int \mathcal{D}[\psi,\overline{\psi}]\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\exp[-S]\Big(i\alpha(x_1) - i\alpha(x_2) - \int d^d x\ \partial_\mu \alpha j^\mu \\ &\qquad + \int d^d x\ i\alpha(x)\left[-2\Delta\overline{\psi}_\uparrow(x)\overline{\psi}_\downarrow(x) + 2\overline{\Delta}\psi_\downarrow(x)\psi_\uparrow(x)\right]\Big) \\ &= \frac{1}{\mathcal{Z}}\int \mathcal{D}[\psi,\overline{\psi}]\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\int d^d x\ i\alpha(x)\Big[\delta(x-x_1) - \delta(x-x_2) - i\partial_\mu j^\mu(x) \\ &\qquad - 2\Delta\overline{\psi}_\uparrow(x)\overline{\psi}_\downarrow(x) + 2\overline{\Delta}\psi_\downarrow(x)\psi_\uparrow(x)\Big]\exp[-S] \\ &= \left\langle T\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\int d^d x\ i\alpha(x)\left(\delta(x-x_1) - \delta(x-x_2) - i\partial_\mu j^\mu(x) - 2\Delta\overline{\psi}_\uparrow(x)\overline{\psi}_\downarrow(x) + 2\overline{\Delta}\psi_\downarrow(x)\psi_\uparrow(x)\right)\right\rangle,\end{aligned} \tag{17}$$

which can be written as

$$\begin{aligned}\left\langle T\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)i\partial_\mu j^\mu(x)\right\rangle &= \delta(x-x_1)\left\langle T\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\right\rangle - \delta(x-x_2)\left\langle T\psi(x_1)_\uparrow\overline{\psi}_\uparrow(x_2)\right\rangle \\ &\quad - 2\Delta\left\langle T\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\overline{\psi}_\uparrow(x)\overline{\psi}_\downarrow(x)\right\rangle + 2\overline{\Delta}\left\langle T\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\psi_\downarrow(x)\psi_\uparrow(x)\right\rangle\end{aligned} \tag{18}$$

By applying the fourier operator $\mathcal{F}\circ = \int_{x,x_1,x_2} e^{ipx}e^{ikx_1}e^{-i(k+p)x_2}$, I find the same result (12) with an additional term:

$$\begin{aligned}p_\mu G(k)\Gamma^\mu(k,k+p)G(k+p) = G(k+p) - G(k) + \mathcal{F}\circ 2\Delta\left\langle T\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\overline{\psi}_\uparrow(x)\overline{\psi}_\downarrow(x)\right\rangle \\ - \mathcal{F}\circ 2\overline{\Delta}\left\langle T\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\psi_\downarrow(x)\psi_\uparrow(x)\right\rangle.\end{aligned} \tag{19}$$

The two 4-point correlators can be evaluated diagrammatically, and are given in Figs.[1-2].

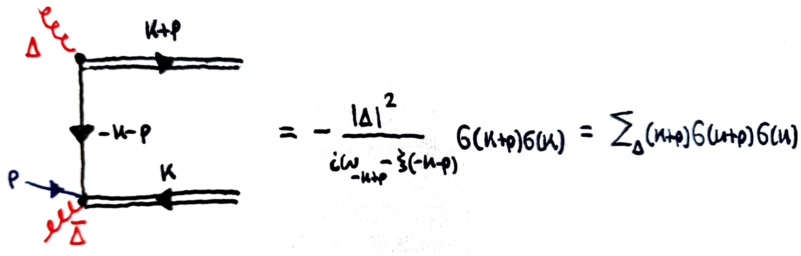


Figure 1: Diagram of $\mathcal{F} \circ \Delta \left\langle T\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\overline{\psi}_\uparrow(x)\overline{\psi}_\downarrow(x)\right\rangle$

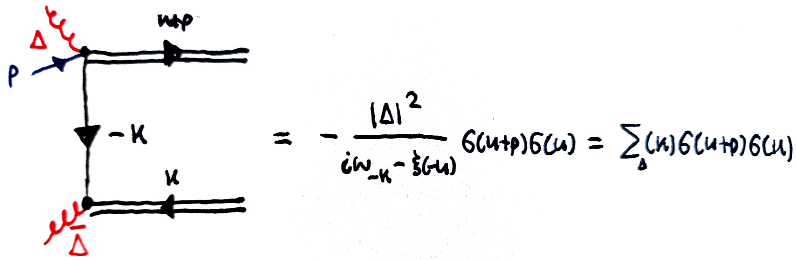


Figure 2: Diagram of $\mathcal{F} \circ \overline{\Delta} \left\langle T\psi_\uparrow(x_1)\overline{\psi}_\uparrow(x_2)\psi_\downarrow(x)\psi_\uparrow(x)\right\rangle$

The choice of $\overline{\psi\psi}$ vs. $\psi\psi$ determines whether or not the p-momentum is inserted on a $\overline{\Delta}$ or Δ vertex, respectively. Note that the intermediate propagator is not re-normalized. This particular combination of a bare propagator and a re-normalized propagator connected by a BCS vertex is also known as an anomalous propagator.

Continuing, Figs.[1-2] allow me to write (19) as

$$\begin{aligned} p_\mu G(k)\Gamma^\mu(k,k+p)G(k+p) &= G(k+p) - G(k) + 2G(k)\left(\Sigma_\Delta(k+p) - \Sigma_\Delta(k)\right)G(k+p) \\ p_\mu \Gamma^\mu(k,k+p) &= -(G^{-1}(k+p) - G^{-1}(k)) + 2\left(\Sigma_\Delta(k+p) - \Sigma_\Delta(k)\right), \end{aligned} \tag{20}$$

whereby taking the limit $\omega \to 0$ then $\mathbf{p} \to 0$ I find the modified Ward identity for a charge-2e superconductor

$$\mathbf{\Gamma}(k,k) = -\partial_{\mathbf{k}} G^{-1}(k) + 2\partial_{\mathbf{k}}\Sigma_\Delta(k) \tag{21}$$

Let's see if this modified Ward identity gives the correct superfluid density. Substituting this into the superfluid density term (3), I find

$$\begin{aligned} \frac{n_s}{m} &= \cancel{\frac{n}{m}} - \int_k \frac{\mathbf{k}}{m}\cdot\left(\cancel{-\partial_{\mathbf{k}}G^{-1}(k)} + 2\partial_{\mathbf{k}}\Sigma_\Delta(k)\right)G(k)^2 \\ &= -2\int_k \frac{\mathbf{k}}{m}\cdot\left(\partial_{\mathbf{k}}\Sigma_\Delta(k)\right)G(k)^2 \\ &= 2\int_k \frac{\mathbf{k}}{m}\cdot\frac{|\Delta|^2\partial_{\mathbf{k}}\xi_{\mathbf{k}}}{(i\omega+\xi_{\mathbf{k}})^2}\frac{(i\omega+\xi_{\mathbf{k}})^2}{(\omega^2+\xi_{\mathbf{k}}^2+|\Delta|^2)^2}, \end{aligned} \tag{22}$$

where I noted that

$$\Sigma_\Delta(k) = \frac{|\Delta|^2}{i\omega+\xi_{\mathbf{k}}}, \qquad G(k) = \frac{1}{i\omega - \xi_{\mathbf{k}} - \Sigma_\Delta(k)}. \tag{23}$$

Since $\xi_{\mathbf{k}} = \frac{1}{2m}(\mathbf{k}+e\mathbf{A})^2 - \mu$, then

$$\begin{aligned} \frac{n_s}{m} &= \frac{2}{m}\int_k \frac{\mathbf{k}}{m}\cdot\frac{|\Delta|^2(\mathbf{k}+e\mathbf{A})}{(\omega^2+\xi_{\mathbf{k}}^2+|\Delta|^2)^2} \\ &= \frac{2}{m}\int_k \frac{|\mathbf{k}|^2}{m}\cdot\frac{|\Delta|^2}{(\omega^2+\xi_{\mathbf{k}}^2+|\Delta|^2)^2}, \end{aligned} \tag{24}$$

where I noted the vector potential term integrates to zero with it being odd in $\mathbf{k}$. For low temperatures at the Fermi surface, introduce $\mu \approx k_F^2/2m$ and a 2d flat density of states:

$$
\begin{aligned}
n_s(T=0) &= 2\mu\nu_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} d\xi \, \frac{|\Delta|^2}{(\omega^2+\xi^2+|\Delta|^2)^2} \\
&= \pi\mu\nu_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{d}{d\omega}\left(\frac{\omega}{\sqrt{\omega^2+|\Delta|^2}}\right).
\end{aligned} \tag{25}
$$

where I absorbed a factor of 2 into making the total density of states ν_0. This is an easy integral to do, leaving

$$
\boxed{n_s(T=0) = \mu\nu_0 = n} \tag{26}
$$

where I noted that $n = 2\mu\nu_0/d$ in d-dimensions, which for the sake of completeness will derive below.

Proof: Note the representations

$$
n = \frac{2}{(2\pi)^d}\int d^d p \, \Theta(\mu-\xi), \qquad \nu = \frac{2}{(2\pi)^d}\int d^d p \, \delta(\mu-\xi), \tag{27}
$$

Both pre-factors of 2 come from the electron spin. Take the dispersion $\xi = p^2/2m$, and so

$$
n = \frac{2\Omega}{(2\pi)^d}\int_0^{\sqrt{2m\mu}} dp \, p^{d-1} = \frac{2\Omega}{(2\pi)^d}\frac{(2m\mu)^{d/2}}{d}, \tag{28}
$$

where Ω is the volume of a $d-1$ sphere. In addition,

$$
\nu = \frac{2\Omega}{(2\pi)^d}\int dp \, p^{d-1}\delta\left(\mu - \frac{p^2}{2m}\right) = \frac{2\Omega}{(2\pi)^d}\int dp \, p^{d-1}\frac{\delta(p-\sqrt{2m\mu})}{\sqrt{2m\mu}/m} = \frac{2\Omega}{(2\pi)^d} m(\sqrt{2m\mu})^{d-2}. \tag{29}
$$

This implies that

$$
2\nu\mu = \frac{2\Omega}{(2\pi)^d}(2m\mu)^{d/2} = nd \iff \boxed{n = 2\nu\mu/d} \tag{30}
$$

as desired.

So, all the electrons have condensed at zero temperature, creating a perfect diamagnet. The wavefunction is completely rigid, and so no paramagnetic current develops at zero temperature in response to an external vector potential.

For non-zero temperatures, I can instead write the Matsubara sum as a CW contour integral over the poles of the Fermi function [1],

$$
n_s(T) = \pi n \oint_{\mathrm{Im}} \frac{dz}{2\pi i} \, f(z)\frac{d}{dz}\left(\frac{z}{\sqrt{|\Delta|^2 - z^2}}\right). \tag{31}
$$

The contour can be deformed to run CCW around the branch cuts along the real axis,

$$
n_s(T) = n \oint_{\mathrm{BC}} \frac{dz}{2i} \, f(z)\frac{d}{dz}\left(\frac{z}{\sqrt{|\Delta|^2 - z^2}}\right), \tag{32}
$$

where the contour BC is depicted in Fig.[3]. Above or below the branch cut, each C_i can be parameterized by either $z = \omega \pm i0^+$. By substituting either of these into the integrand (33), it's clear that the real component of the term in parentheses is proportional to 0^+ and vanishes, leaving just

$$
n_s(T) = n \oint_{\mathrm{BC}} \frac{dz}{2} \, f(z)\frac{d}{dz}\mathrm{Im}\left(\frac{z}{\sqrt{|\Delta|^2 - z^2}}\right). \tag{33}
$$

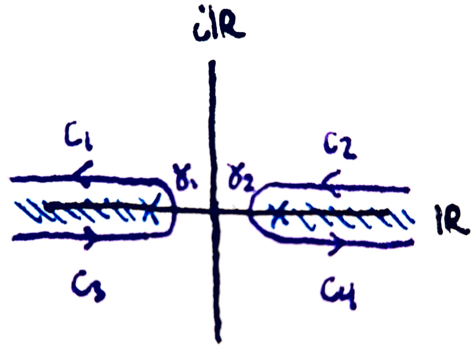


Figure 3: Contour integral for the computation of (33). The branch points are $\pm|\Delta|$.

By parameterizing the paths γ_1, γ_2 infinitesimally around the branch point, it can be shown that they do not contribute to $n_s(T)$. The paths C_i likely do contribute:

$$\begin{aligned}\frac{n_s(T)}{n} = &-\frac{1}{2}\int_{-\infty}^{-|\Delta|} d\omega\ f(\omega)\frac{d}{d\omega}\mathrm{Im}\left(\frac{z}{\sqrt{|\Delta|^2-z^2}}\right)_{\omega+i0^+} + \frac{1}{2}\int_{-\infty}^{-|\Delta|} d\omega\ f(\omega)\frac{d}{d\omega}\mathrm{Im}\left(\frac{z}{\sqrt{|\Delta|^2-z^2}}\right)_{\omega-i0^+} \\ &-\frac{1}{2}\int_{|\Delta|}^{\infty} d\omega\ f(\omega)\frac{d}{d\omega}\mathrm{Im}\left(\frac{z}{\sqrt{|\Delta|^2-z^2}}\right)_{\omega+i0^+} + \frac{1}{2}\int_{|\Delta|}^{\infty} d\omega\ f(\omega)\frac{d}{d\omega}\mathrm{Im}\left(\frac{z}{\sqrt{|\Delta|^2-z^2}}\right)_{\omega-i0^+},\end{aligned} \tag{34}$$

where it suffices to compute

$$\begin{aligned}\mathrm{Im}\left(\frac{z}{\sqrt{|\Delta|^2-z^2}}\right)_{\omega\pm i0^+} = \mathrm{Im}\left(\frac{\omega\pm i0^+}{\sqrt{|\Delta|^2-(\omega\pm i0^+)^2}}\right) &= \mathrm{Im}\left(\frac{\omega}{\sqrt{-(\omega^2-|\Delta|^2)\mp i\,\mathrm{sgn}(\omega)0^+}}\right) \\ &= \mathrm{Im}\left(\mp\frac{|\omega|}{i\sqrt{\omega^2-|\Delta|^2}}\Theta(\omega^2-|\Delta|^2)\right) \\ &= \pm\frac{|\omega|}{\sqrt{\omega^2-|\Delta|^2}}\Theta(\omega^2-|\Delta|^2).\end{aligned} \tag{35}$$

In light of (34), the integrals in each row add:

$$\frac{n_s(T)}{n} = -\left(\int_{-\infty}^{-|\Delta|} d\omega + \int_{|\Delta|}^{\infty} d\omega\right)\ f(\omega)\frac{d}{d\omega}\left(\frac{|\omega|}{\sqrt{\omega^2-|\Delta|^2}}\right). \tag{36}$$

Continuing with an integration by parts gives me the result

$$\begin{aligned}\frac{n_s(T)}{n} = &-f(\omega)\frac{d}{d\omega}\left(\frac{|\omega|}{\sqrt{\omega^2-|\Delta|^2}}\right)\Big|_{-\infty}^{-|\Delta|} - f(\omega)\frac{d}{d\omega}\left(\frac{|\omega|}{\sqrt{\omega^2-|\Delta|^2}}\right)\Big|_{|\Delta|}^{\infty} \\ &-\left(\int_{-\infty}^{-|\Delta|} d\omega + \int_{|\Delta|}^{\infty} d\omega\right)\left(-\frac{df(\omega)}{d\omega}\right)\left(\frac{|\omega|}{\sqrt{\omega^2-|\Delta|^2}}\right),\end{aligned} \tag{37}$$

$$\boxed{\frac{n_s(T)}{n} = 1 - 2\int_{|\Delta|(T)}^{\infty} d\omega\ \left(-\frac{df(\omega)}{d\omega}\right)\left(\frac{\omega}{\sqrt{\omega^2-|\Delta|^2}}\right)} \tag{38}$$

for $n_s(T)$. The second term reduces the fraction of condensed particles, and does so by the thermal depopulation of the condensate into quasiparticles, as it represents the thermal average of the quasiparticle density of states $\nu_{\mathrm{qp}} = \nu_0\frac{E}{\sqrt{E^2-|\Delta|^2}}$ [1].

References

[1] Piers Coleman. *Introduction to Many-Body Physics*. Cambridge University Press, 2015.

[2] Nikolay V. Gnezdilov and Yuxuan Wang. Solvable model for a charge-4e superconductor. *arXiv preprint*, 2021.

[3] Steven Weinberg. *The Quantum Theory of Fields, Volume I: Foundations*. Cambridge University Press, reprint edition, 2005.