Poor Man's Scaling of the Kondo Model

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Preface: These are notes from various references [1, 4, 5] on the poor man's scaling, or renormalization group analysis, of the single impurity Kondo model. The textbook by Piers Coleman [3] was very helpful, in addition to that of [7, 6, 2].

The Kondo model is most often presented as

$$H_K = \sum_k \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + J \sum_{k,k'} S^a_d \cdot c^\dagger_{k\alpha} \sigma^a_{\alpha\beta} c_{k'\beta}, \tag{1}$$

and represents an antiferromagnetic coupling J between a single fixed magnetic moment (spin) $S^a_d = 1/2$ with a sea of (delocalized) conduction electrons $c^\dagger_{k\alpha}$ with dispersion ε_k limited to some energy scale (band-width) D. This interaction was devised by Jun Kondo to explain the increase in resistivity $\rho(T)$ as $T \to 0$ for metals containing magnetic impurities, in contrast to other mechanisms (electron-electron and phonon interaction) which tend to lower the resistivity with decreases in temperature. He found that to second order in Kondo coupling J, the resistivity had the form $\rho(T) \sim \left(J + \nu J^2 \ln\left(\frac{D}{T}\right) + \mathcal{O}(J^3)\right)^2$, which shows that as T decreases, the resistivity increases logarithmically. Two questions arise: (1) what is the behavior of the resistivity around the Kondo temperature $T \sim T_K \equiv D\exp(-1/\nu J)$, in which the 1st and 2nd order terms are together negligible, and (2) is the resistivity truly divergent as $T \to 0$?

The key idea in determining this low temperature behavior is the concept of renormalization group. In particular, I will be following a "Poor Man's Scaling" approach done by Anderson in 1970. This scheme involves the computation of Feynman diagrams which contribute to the renormalization of the Kondo coupling J_K, i.e., the first and second order contributions to the T-matrix. Such an approach is slightly complicated because the fixed d-spin is no longer a free fermion, and there is no Wick's theorem for the correlations between spin operators. This can be dealt with using the Abrikosov pseudo-fermion representation of the spin operators,

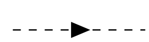
$$S^a_d = f^\dagger_\alpha \frac{\sigma^a_{\alpha\beta}}{2} f_\beta, \tag{2}$$

where since this representation expands the Hilbert space, I must impose the additional constraint $f^\dagger_\alpha f_\alpha = 1$ which eliminates empty and doubly occupied states. Popov and Fedotov have a clever way of implementing this constraint via a complex chemical potential $\mu = -i\pi T/2$, which affects the partition function in the following way,

$$\begin{aligned} \mathcal{Z} = \mathrm{Tr}\left[e^{-\beta\left(H + \frac{i\pi T}{2}(n_f - 1)\right)}\right] &= e^{i\pi/2}\mathcal{Z}(f^0) + \mathcal{Z}(f^1) + e^{-i\pi/2}\mathcal{Z}(f^2) \\ &= \mathcal{Z}(f^1), \end{aligned} \tag{3}$$

where I noted that $\mathcal{Z}(f^0)$ and $\mathcal{Z}(f^2)$ have equal weight ($S_f = 0$ in both) and the pre-factors are opposite $e^{i\pi/2} = -e^{i\pi/2} = i$, leaving only the $\mathcal{Z}(f^1)$ contribution. Note that the expansion of the partition function into these sectors is justified since $[H, n_f]$. With this approach, the f-propagators are

$$G_f(i\omega_n) =$$

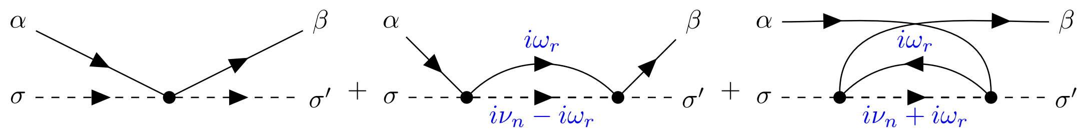


$$= \frac{1}{i\omega_n + \mu} = \frac{1}{i\omega_n - i\pi T/2}, \tag{4}$$

representative of a shifted Matsubara frequency $\omega_n \to \omega_n - \pi T/2$.

There are three diagrams which contribute to the renormalization of J to second order,

$$-\Gamma =$$



$$\equiv \phi + \Gamma_1 + \Gamma_2, \tag{5}$$

each of which can be evaluated using the Feynman rules. Note that I am taking the total energy in the particle-particle and particle-hole channels as $i\nu_n$, and recall that that the c-electron propagator has the form

$$G_c(i\omega_n, \vec{k}) = \quad \longrightarrow \quad = \frac{1}{i\omega_n - \varepsilon_k}. \tag{6}$$

Each of these diagrams are straight-forward,

$$\begin{aligned}
\phi &= -J\sigma^a_{\beta\alpha} S^a_{\sigma'\sigma}, \qquad \left(\vec{S}_{\sigma'\sigma} \equiv \frac{\vec{\sigma}_{\sigma'\sigma}}{2}\right) \\
\Gamma_1 &= J^2(\sigma^b\sigma^a)_{\beta\alpha}(S^bS^a)_{\sigma'\sigma}\psi_1(i\nu_n) \\
\Gamma_2 &= -J^2(\sigma^a\sigma^b)_{\beta\alpha}(S^bS^a)_{\sigma'\sigma}\psi_2(i\nu_n)
\end{aligned} \tag{7}$$

where I have redefined the spin operator without the external fermion legs, as well as included a minus sign for the crossing of fermion lines in Γ_2. The bubbles ψ_1 and ψ_2 have the form

$$\psi_1(i\nu) = T\sum_{i\omega_r,k} \frac{1}{i\omega_r - \varepsilon_k}\frac{1}{i\nu_n - i\omega_r + \mu}, \qquad \psi_2(i\nu_n) = -T\sum_{i\omega_r,k}\frac{1}{i\omega_r - \varepsilon_k}\frac{1}{i\nu_n + i\omega_r + \mu}, \tag{8}$$

which through the process of relabelling $\omega_r \to -\omega_r$ and using a particle-hole symmetric density of states, $\varepsilon_k \to -\varepsilon_k$, shows that both bubbles have equal weight: $\psi_1(i\nu_n) = \psi_2(i\nu_n) \equiv \psi(i\nu_n)$. It is straightforward to recast $\psi(i\nu_n)$ into a contour integral[1]

$$\begin{aligned}
\psi(i\nu_n) &= -\sum_k \left[\oint_{z_0=\varepsilon_k} \frac{dz}{2\pi i}\, f(z) + \oint_{z_0=-i\nu_n-\mu} \frac{dz}{2\pi i}\, f(z)\right] \frac{1}{z-\varepsilon_k}\frac{1}{z+i\nu_n+\mu} \\
&= -\sum_k \left[\oint_{z_0=\varepsilon_k} \frac{dz}{2\pi i}\, \frac{f(z)}{z+i\nu_n+\mu}\cdot\frac{1}{z-\varepsilon_k} + \oint_{z_0=-i\nu_n-\mu}\frac{dz}{2\pi i}\,\frac{f(z)}{z-\varepsilon_k}\cdot\frac{1}{z+i\nu_n+\mu}\right] \\
&= -\sum_k \left[\frac{f(\varepsilon_k)}{i\nu_n+\mu+\varepsilon_k} - \frac{f(-i\nu_n-\mu)}{i\nu_n+\mu+\varepsilon_k}\right] \\
&= \sum_k \frac{f(-\mu) - f(\varepsilon_k)}{i\nu_n + \mu + \epsilon_k}.
\end{aligned} \tag{9}$$

where $f(z)$ is the Fermi-Dirac distribution. Note that the third equality is a consequence of Cauchy's Integral Formula, and I used the periodicity of the Fermi-Dirac distribution,

$$f(-i\nu_n - \mu) = \frac{1}{e^{-i\beta\nu_n}e^{-\beta\mu}+1} = \frac{1}{e^{-2\pi i n}e^{-\beta\mu}+1} = f(-\mu). \tag{10}$$

Transforming the momentum summation and using $\mu = -i\pi T/2$,

$$\psi(i\nu_n) = \int d\varepsilon\,\rho(\varepsilon)\frac{f(-\mu) - f(\varepsilon)}{i\nu_n - i\pi T/2 + \epsilon}, \tag{11}$$

[1] There are poles at each $i\omega_r$ on the imaginary axis, in addition to $z_1 = \epsilon_k$ and $z_2 = -i\nu_n - \mu$. Since the integrand vanishes faster than $1/|z|$ over the whole contour, Jordan's Lemma is satisfied and the only nonzero contributions are the CCW paths around the two poles z_1, z_2.

which is, in general, difficult to solve at finite temperature.[2] This integral can be evaluated for a Lorentzian density of states $\rho(\varepsilon) = \rho D^2/(D^2+\varepsilon^2)$ [3],

$$\begin{aligned}\psi(i\nu_n) &= \rho_0 \left(\ln\left(\frac{D}{2\pi T}\right) - \psi\left(\frac{1}{2} + \frac{(i\nu_n - i\pi T/2)\,\beta}{2\pi i}\right) - \frac{i\pi}{2}\tanh\left(\frac{i\pi}{4}\right)\right) \\ &= \rho_0\left(\ln\left(\frac{De^{\pi/2}}{2\pi T}\right) - \psi\left(\frac{1}{2} + \frac{\nu_n - \pi T/2}{2\pi T}\right)\right)\end{aligned} \tag{12}$$

where I used the fact that $\tanh(i\pi/4) = i\tan(\pi/4) = i$, then brought the bare $\pi/2$ factor inside the logarithm. Note that this evaluation is valid iff $\nu_n - \pi T/2 > 0$, and the other case is determined via complex conjugation [3]. Since (12) is real, it must depend on the norm $|\nu_n - \pi T/2|$. Nevertheless, I seek to evaluate this bubble on the Fermi surface, $\nu_n = \pi T/2$,

$$\psi(i\pi T/2) = \rho_0 \ln\left(\frac{De^{\pi/2-\psi(1/2)}}{2\pi T}\right). \tag{13}$$

Summing all the diagrams, I find

$$\begin{aligned}-\Gamma &= -J\sigma^a_{\beta\alpha}S^a_{\sigma'\sigma} + \rho_0 J^2 \ln\left(\frac{De^{\pi/2-\psi(1/2)}}{2\pi T}\right)\left(\sigma^b\sigma^a - \sigma^a\sigma^b\right)_{\beta\alpha}\left(S^bS^a\right)_{\sigma'\sigma} \\ &= -J\sigma^a_{\beta\alpha}S^a_{\sigma'\sigma} + \rho_0 J^2 \ln\left(\frac{De^{\pi/2-\psi(1/2)}}{2\pi T}\right)\left(2i\epsilon^{bac}\sigma^c_{\beta\alpha}\right)\left(\cancel{\frac{1}{4}\delta^{ba}I_{\sigma'\sigma}} + \frac{i}{2}\epsilon^{bad}S^d_{\sigma'\sigma}\right) \\ &= -J\sigma^a_{\beta\alpha}S^a_{\sigma'\sigma} - \rho_0 J^2 \ln\left(\frac{De^{\pi/2-\psi(1/2)}}{2\pi T}\right)\epsilon^{bac}\epsilon^{bad}\sigma^c_{\beta\alpha}S^d_{\sigma'\sigma} \\ &= -\left(J + 2\rho_0 J^2 \ln\left(\frac{De^{\pi/2-\psi(1/2)}}{2\pi T}\right)\right)\sigma^a_{\beta\alpha}S^a_{\sigma'\sigma} \\ &\equiv -\left(J+\delta J\right)\sigma^a_{\beta\alpha}S^a_{\sigma'\sigma} \implies \delta J = 2\rho_0 J^2 \ln\left(\frac{De^{\pi/2-\psi(1/2)}}{2\pi T}\right),\end{aligned} \tag{14}$$

which gives the variation (renormalization) of the Kondo coupling to second-order. Since RG only provides the leading logarithmic accuracy, it is common to drop the dimensionless constants inside the logarithm, leaving

$$\delta J = 2\rho_0 J^2 \ln\left(\frac{D}{T}\right) \iff \beta(J) = \frac{dJ}{d\ln(D)} = -2\rho_0 J^2, \tag{15}$$

as the RG beta function. Note that (15) implies J is marginal.[3] Integrating this relation, I find

$$\int_J^{J_0} \frac{dJ'}{J'^2} = -2\rho_0 \int_{\ln(T)}^{\ln(D)} d\ln(D') \implies \frac{1}{J} - \frac{1}{J_0} = -2\rho_0 \ln\left(\frac{D}{T}\right), \tag{16}$$

or simply

$$\boxed{J(T) = \frac{J_0}{1 - 2\rho_0 J_0 \ln\left(\frac{D}{T}\right)}} \tag{17}$$

as the effective Kondo coupling constant.

If $J_0 > 0$ (antiferromagnetic), then around the Kondo temperature

$$T_K \sim De^{-1/2\rho_0 J_0}, \tag{18}$$

the system scales to strong coupling, i.e. J becomes quite large.[4] So, as the temperature is lowered to the Kondo temperature, perturbation theory breaks down. Regardless, in terms of T_K, J is no longer dependent on the initial cutoff

[2] A bound can be found easily via a flat density of states at $T = 0$.

[3] Marginal: The beta function is not linearly dependent on its argument.

[4] Wilson showed using a numerical renormalization group that the RG flow is indeed $J \to \infty$ to all orders, so the prediction of (17) valid to second order is true.

energy scale, implying that the low energy behavior of the Kondo model is certainly independent of any high-energy physics. There is now only a single, dynamically generated scale, the Kondo temperature. Physically, this strong coupling fixed point has the magnetic impurity screened by the conduction electrons, removing the internal degrees of freedom to form a Fermi liquid.

In the case that $J_0 < 0$ (ferromagnetic), then $J \to 0$ as $T \to 0$. This fixed point $T = 0$, comprising a decoupled moment, is thus attractive for the FM case and repulsive for the AFM case.

Anisotropic Coupling: This argument can be generalized to an anisotropic Kondo model. In particular, the diagrams (5) have the same form with a redefinition of the Kondo coupling depending on the particular spin structure. Note that Γ_1 and Γ_2 make it so that terms with $a = b$ do not evolve the coupling, and so only mixing between spins occur. Take J_x, in which the diagram sum (14) becomes

$$\begin{aligned} -\Gamma &= -J_x \sigma^x S^x + \rho_0 J_y J_z \ln\left(\frac{D}{T}\right) \left(\sigma^y \sigma^z - \sigma^z \sigma^y\right) \left(S^y S^z\right) \\ &= -\left[J_x + 2\rho_0 J_y J_z \ln\left(\frac{D}{T}\right)\right] \sigma^x S^x \implies \delta J_x = 2\rho_0 J_y J_z \ln\left(\frac{D}{T}\right). \end{aligned} \tag{19}$$

where I've dropped the dimensionless constants inside the logarithm and took only the y, z term at second order since $S^y S^z = i S^x$. The other Kondo couplings follow similarly,

$$\delta J_y = 2\rho_0 J_x J_z \ln\left(\frac{D}{T}\right), \qquad \delta J_z = 2\rho_0 J_x J_y \ln\left(\frac{D}{T}\right). \tag{20}$$

In the case that $J_x = J_y \equiv J_{xy}$, the corresponding beta functions are

$$\boxed{\frac{dJ_{xy}}{d\ln(D)} = -2\rho_0 J_{xy} J_z} \qquad \boxed{\frac{dJ_z}{d\ln(D)} = -2\rho_0 J_{xy}^2} \tag{21}$$

which are mathematically equivalent to the beta functions of the Berezinskii-Kosterlitz-Thouless (BKT) transition, which is a phase transition of the 2d XY-model. Through some manipulations,

$$J_{xy} \frac{dJ_{xy}}{d\ln(D)} = J_z \frac{dJ_z}{d\ln(D)} \implies \frac{d}{d\ln(D)} \left(J_z^2 - J_{xy}^2\right) = 0, \tag{22}$$

and hence $J_z^2 - J_{xy}^2 = \text{const.}$, giving rise to hyperbolic curves in the scaling trajectories. Denoting the constant as c^2, the flow of J_z can be computed directly from (21),

$$\int_{J_z}^{J_{0,z}} \frac{dJ_z}{c^2 - J_z^2} = 2\rho_0 \int_{\ln(T)}^{\ln(D_0)} d\ln(D) \implies J_z = \frac{c(J_{0,z} - c \tanh\left[2\rho_0 c \ln\left(\frac{D_0}{T}\right)\right])}{c - J_{0,z} \tanh\left[2\rho_0 c \ln\left(\frac{D_0}{T}\right)\right]}. \tag{23}$$

Assume that the initial J_z coupling is antiferromagnetic. Then, as the temperature is lowered to

$$T_c = D_0 \exp\left(-\frac{1}{2\rho_0 c} \tanh^{-1}\left(\frac{c}{J_{0,z}}\right)\right), \tag{24}$$

the system approaches strong coupling. For an initial ferromagnetic coupling $J_{0,z} = -|J_{0,z}|$ there are no singularities and the coupling approaches $J_z = -c$ at low temperature (at this fixed point, $J_{xy}^2 = J_z^2 - c^2 = 0$).

References

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