# Exact Solution to Maxwell's Equations Given Sources

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**Preface:** This calculation was shown to me in the course PHY6346 Graduate Electrodynamics 1 at the University of Florida, taught by Professor Richard Woodard.

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A spatial Fourier transformation of Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}, \qquad \nabla \times \mathbf{B} - \epsilon\mu\dot{\mathbf{E}} = \mu\mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0, \tag{1}$$

allows for the theory to be uncoupled into a collection of simple harmonic oscillators. To see this I will work instead with the two equations describing EM waves in media, which are easy to compute by taking curls of the vector Maxwell equations (1), using a vector identity, and substituting in the according scalar Maxwell equations:

$$\left[\nabla^2 - \epsilon\mu\partial_t^2\right]\mathbf{E} = \frac{1}{\epsilon}\nabla\rho + \mu\dot{\mathbf{J}}, \qquad\qquad \left[\nabla^2 - \epsilon\mu\partial_t^2\right]\mathbf{B} = -\mu\nabla\times\mathbf{J}. \tag{2}$$

I will define the spatial Fourier transform as

$$\tilde{f}(\mathbf{k}) = \int d^3r \, f(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}} \iff f(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} \, \tilde{f}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{3}$$

from which follows the replacement $\widetilde{\nabla f} = -i\mathbf{k}\tilde{f}$. Now, (2) may be written

$$\left[-k^2 - \epsilon\mu\partial_t^2\right]\widetilde{\mathbf{E}} = -\frac{i}{\epsilon}\mathbf{k}\tilde{\rho} + \mu\dot{\widetilde{\mathbf{J}}} \iff \left[\partial_t^2 + v^2k^2\right]\widetilde{\mathbf{E}} = \frac{1}{\epsilon}\left[iv^2\mathbf{k}\tilde{\rho} - \dot{\widetilde{\mathbf{J}}}\right],$$

$$\left[-k^2 - \epsilon\mu\partial_t^2\right]\widetilde{\mathbf{B}} = i\mu\mathbf{k}\times\widetilde{\mathbf{J}} \iff \left[\partial_t^2 + v^2k^2\right]\widetilde{\mathbf{B}} = -\frac{i}{\epsilon}\mathbf{k}\times\widetilde{\mathbf{J}}, \tag{4}$$

where I have defined some quantity $v = (\epsilon\mu)^{-1/2}$ with the dimensions of speed. These equations describe driven harmonic oscillators with, which can be further simplified from a Green's function approach.

## Aside - SHO Green's Function

Let $[d_t^2 + \omega^2]G(t;t') = \delta(t - t')$ with a retarded boundary condition $G(t;t' > t) = 0$. It's natural to take the ansatz $G(t;t') = \Theta(t - t')\left[A\cos(\omega t) + B\sin(\omega t)\right]$ to enforce the boundary condition and homogeneous solution, and I find by taking two time-derivatives that

$$[d_t^2 + \omega^2]G(t;t') = \delta'(t - t')\left[A\cos(\omega t') + B\sin(\omega t')\right] + \delta(t - t')\left[-\omega A\sin(\omega t') + \omega B\cos(\omega t')\right], \tag{5}$$

The RHS must be $\delta(t - t')$, leading to two equations for the undetermined coefficients:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{\omega}\begin{pmatrix} \omega\cos(\omega t') & -\sin(\omega t') \\ \omega\sin(\omega t') & \cos(\omega t') \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{6}$$

and hence

$$G(t;t') = \frac{\Theta(t - t')}{\omega}\left[-\sin(\omega t')\cos(\omega t) + \cos(\omega t')\sin(\omega t)\right] = \frac{\Theta(t - t')}{\omega}\sin(\omega(t - t')) \tag{7}$$

is the Green's function for the SHO.

For a general source term $S(t)$, such that $[d_t^2 + \omega^2]q(t) = S(t)$, then $q(t)$ can be written

$$q(t) = q_0 \cos(\omega t) + \frac{\dot{q}_0}{\omega} \sin(\omega t) + \int_{-\infty}^{\infty} dt' G(t; t') S(t')$$

$$= q_0 \cos(\omega t) + \frac{\dot{q}_0}{\omega} \sin(\omega t) + \int_0^t dt' \frac{\sin(\omega(t - t'))}{\omega} S(t').$$

(8)

where the first two terms are the homogeneous solution.

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From (8), the momentum space fields (4) can be written succinctly as

$$\widetilde{\mathbf{E}}(t, \mathbf{k}) = \widetilde{\mathbf{E}}_0(\mathbf{k}) \cos(vkt) + \frac{\dot{\widetilde{\mathbf{E}}}_0(\mathbf{k})}{vk} \sin(vkt) + \frac{1}{\epsilon} \int_0^t dt' \frac{\sin(vk(t - t'))}{vk} \left[ iv^2 \mathbf{k} \widetilde{\rho}(t', \mathbf{k}) - \dot{\widetilde{\mathbf{J}}}(t', \mathbf{k}) \right],$$

$$\widetilde{\mathbf{B}}(t, \mathbf{k}) = \widetilde{\mathbf{B}}_0(\mathbf{k}) \cos(vkt) + \frac{\dot{\widetilde{\mathbf{B}}}_0(\mathbf{k})}{vk} \sin(vkt) - \frac{1}{\epsilon} \int_0^t dt' \frac{\sin(vk(t - t'))}{vk} \left[ i\mathbf{k} \times \widetilde{\mathbf{J}}(t', \mathbf{k}) \right].$$

(9)

Maxwell's equations (1) can be used to write the initial value data in terms of only $\widetilde{\mathbf{E}}_0$ and $\widetilde{\mathbf{B}}_0$:

$$\nabla \times \mathbf{B} - \epsilon\mu\dot{\mathbf{E}} = \mu\mathbf{J} \iff -i\mathbf{k} \times \widetilde{\mathbf{B}} - \epsilon\mu\dot{\widetilde{\mathbf{E}}} = \mu\widetilde{\mathbf{J}} \iff \dot{\widetilde{\mathbf{E}}}_0(\mathbf{k}) = -iv^2 \mathbf{k} \times \widetilde{\mathbf{B}}_0(\mathbf{k}) - \frac{1}{\epsilon} \widetilde{\mathbf{J}}_0(\mathbf{k}),$$

$$\text{and} \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}} \iff \dot{\widetilde{\mathbf{B}}}_0(\mathbf{k}) = i\mathbf{k} \times \widetilde{\mathbf{E}}_0,$$

(10)

which allows (9) to be written as

$$\widetilde{\mathbf{E}}(t, \mathbf{k}) = \widetilde{\mathbf{E}}_0(\mathbf{k}) \cos(vkt) - iv\hat{k} \times \widetilde{\mathbf{B}}_0(\mathbf{k}) \sin(vkt) - \frac{1}{\epsilon vk} \widetilde{\mathbf{J}}_0(\mathbf{k}) \sin(vkt)$$

$$+ \frac{1}{\epsilon} \int_0^t dt' \frac{\sin(vk(t - t'))}{vk} \left[ iv^2 \mathbf{k} \widetilde{\rho}(t', \mathbf{k}) - \dot{\widetilde{\mathbf{J}}}(t', \mathbf{k}) \right],$$

$$\widetilde{\mathbf{B}}(t, \mathbf{k}) = \widetilde{\mathbf{B}}_0(\mathbf{k}) \cos(vkt) + \frac{i}{v} \hat{k} \times \widetilde{\mathbf{E}}_0(\mathbf{k}) \sin(vkt) - \frac{1}{\epsilon} \int_0^t dt' \frac{\sin(vk(t - t'))}{vk} \left[ i\mathbf{k} \times \widetilde{\mathbf{J}}(t', \mathbf{k}) \right].$$

(11)

By noting that

$$\frac{1}{\epsilon} \int_0^t dt' \frac{\sin(vk(t - t'))}{vk} \left[ -\dot{\widetilde{\mathbf{J}}}(t', \mathbf{k}) \right] = \frac{1}{\epsilon vk} \widetilde{\mathbf{J}}_0(\mathbf{k}) \sin(vkt) - \frac{1}{\epsilon} \int_0^t dt' \cos(vk(t - t')) \widetilde{\mathbf{J}}(t', \mathbf{k})$$

(12)

through an integration by parts, I can write (11) as

$$\widetilde{\mathbf{E}}(t, \mathbf{k}) = \widetilde{\mathbf{E}}_0(\mathbf{k}) \cos(vkt) - iv\hat{k} \times \widetilde{\mathbf{B}}_0(\mathbf{k}) \sin(vkt)$$

$$+ \frac{1}{\epsilon} \int_0^t dt' \left[ \sin(vk(t - t')) \cdot iv\hat{k} \widetilde{\rho}(t', \mathbf{k}) - \cos(vk(t - t')) \widetilde{\mathbf{J}}(t', \mathbf{k}) \right],$$

$$\widetilde{\mathbf{B}}(t, \mathbf{k}) = \widetilde{\mathbf{B}}_0(\mathbf{k}) \cos(vkt) + \frac{i}{v} \hat{k} \times \widetilde{\mathbf{E}}_0(\mathbf{k}) \sin(vkt) - \frac{1}{\epsilon} \int_0^t dt' \sin(vk(t - t')) \cdot \frac{i}{v} \hat{k} \times \widetilde{\mathbf{J}}(t', \mathbf{k}).$$

(13)

which are the full electric and magnetic fields in momentum space, determined solely by initial values and sources.

It is sometimes useful to rewrite the solution for $\widetilde{E}(t,\mathbf{k})$ in the following manner; since

$$\sin(vk(t-t')) = \partial_{t'}\left[(1/vk)\cos(vk(t-t'))\right], \tag{14}$$

then

$$\frac{1}{\epsilon}\sin(vk(t-t'))\cdot iv\hat{k}\widetilde{\rho} = \partial_{t'}\left[\frac{i\hat{k}}{\epsilon k}\cos(vk(t-t'))\widetilde{\rho}\right] - \frac{i\hat{k}}{\epsilon k}\cos(vk(t-t'))\dot{\widetilde{\rho}}$$

$$= \partial_{t'}\left[\frac{i\hat{k}}{\epsilon k}\cos(vk(t-t'))\widetilde{\rho}\right] + \frac{1}{\epsilon}\cos(vk(t-t'))\hat{k}(\hat{k}\cdot\widetilde{\mathbf{J}}), \tag{15}$$

where in the last step I used a SFT on the continuity equation: $\nabla\cdot\mathbf{J} + \dot{\rho} = 0 \implies \dot{\widetilde{\rho}} = i\mathbf{k}\cdot\widetilde{\mathbf{J}}$. It follows that the integral in $\widetilde{\mathbf{E}}(t,\mathbf{k})$ (13) can be re-expressed as

$$\frac{1}{\epsilon}\int_0^t dt'\left[\sin(vk(t-t'))\cdot iv\hat{k}\widetilde{\rho}(t',\mathbf{k}) - \cos(vk(t-t'))\widetilde{\mathbf{J}}(t',\mathbf{k})\right]$$

$$=$$

$$\frac{i\hat{k}}{\epsilon k}\left[\widetilde{\rho}(t,\mathbf{k}) - \cos(vkt)\widetilde{\rho}_0(\mathbf{k})\right] - \frac{1}{\epsilon}\int_0^t dt'\cos(vk(t-t'))\left[\widetilde{\mathbf{J}}(t',\mathbf{k}) - \hat{k}(\hat{k}\cdot\widetilde{\mathbf{J}}(t',\mathbf{k}))\right] \tag{16}$$

$$=$$

$$\frac{i\hat{k}}{\epsilon k}\widetilde{\rho}(t,\mathbf{k}) - \hat{k}(\hat{k}\cdot\widetilde{\mathbf{E}}_0(\mathbf{k}))\cos(vkt) - \frac{1}{\epsilon}\int_0^t dt'\cos(vk(t-t'))\widetilde{\mathbf{J}}_T(t',\mathbf{k})$$

where I used a SFT of Gauss' Law, $\nabla\cdot\mathbf{E} = \rho/\epsilon \implies \widetilde{\rho} = -i\epsilon\mathbf{k}\cdot\widetilde{\mathbf{E}}$, and recognized the combination $\widetilde{\mathbf{J}}_T \equiv \widetilde{\mathbf{J}} - \hat{k}(\hat{k}\cdot\widetilde{\mathbf{J}})$ as the transverse current. Therefore, the electric field term (13) can be written

$$\widetilde{\mathbf{E}}(t,\mathbf{k}) = \left(\widetilde{\mathbf{E}}_0(\mathbf{k}) - \hat{k}(\hat{k}\cdot\widetilde{\mathbf{E}}_0(\mathbf{k}))\right)\cos(vkt) - iv\hat{k}\times\widetilde{\mathbf{B}}_0(\mathbf{k})\sin(vkt) + \frac{i\hat{k}}{\epsilon k}\widetilde{\rho}(t,\mathbf{k})$$

$$- \frac{1}{\epsilon}\int_0^t dt'\cos(vk(t-t'))\widetilde{\mathbf{J}}_T(t',\mathbf{k}), \tag{17}$$

where $\widetilde{\mathbf{E}}_{0,T} = \widetilde{\mathbf{E}}_0 - \hat{k}(\hat{k}\cdot\widetilde{\mathbf{E}}_0)$ is the transverse component of the initial electric field and $\frac{i\hat{k}}{\epsilon k}\widetilde{\rho}(t,\mathbf{k})$ is the instantaneous electric field, since

$$\mathbf{E}_{\text{inst.}} = \frac{1}{\epsilon}\int\frac{d^3k}{(2\pi)^3}\frac{i\mathbf{k}}{k^2}\widetilde{\rho}(t',\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}} = -\frac{1}{\epsilon}\nabla\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\widetilde{\rho}(t',\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}$$

$$= \frac{1}{\epsilon}\nabla\left(\nabla^{-2}\int\frac{d^3k}{(2\pi)^3}\widetilde{\rho}(t',\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}\right) \iff \nabla^2\mathbf{E}_{\text{inst.}} = \frac{1}{\epsilon}\nabla\rho \tag{18}$$

which is indeed the equation satisfied by the instantaneous electric field (c.f. vanishing of $\partial_t^2$ and $\dot{\mathbf{J}}$ in (2)).