Zeta Regularization and the Casimir Effect

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Jan 2024

Preface: These notes are my solution to problem 8.6 of Eduardo Fradkin's textbook [1]. I take a real scalar field in a 1+1 Euclidean spacetime and evaluate the ground state energy density through Riemann zeta function regularization. I find a correction which results in an effective attractive force on the walls of the system by the zero-motion of the field, known as the Casimir effect.

Consider a free scalar field $\phi(x)$ in 1+1 dimensions, where $x = (t, x)$. The Lagrangian density has the following form,

$$\mathcal{L} = \frac{1}{2}\partial_\mu(x)\partial^\mu(x) - \frac{1}{2}m^2\phi(x)^2. \tag{1}$$

Let the total length of the system along the spatial coordinate by L, and assume periodic boundary conditions $\phi(x,t) = \phi(x+L,t)\ \forall t$. It is simple to show that the classical ground state energy for this system is zero, support by the fact that the corresponding Hamiltonian is positive definite.

1 Partition Function

To compute the quantum correction to this energy density, I begin with the vacuum persistence amplitude, or generating functional, of this theory,

$$Z[J] = \int \mathcal{D}\phi\ e^{iS[\phi,J]}. \tag{2}$$

For the equivalent Euclidean theory, I will perform a Wick rotation $t = -i\tau$, giving

$$\begin{aligned}\mathcal{L}[\phi,J] = \frac{1}{2}\left(\partial_0\phi\right)^2 - \frac{1}{2}\left(\partial_x\phi\right)^2 - \frac{1}{2}m^2\phi(x)^2 + J\phi \mapsto -\frac{1}{2}\left(\partial_\tau\phi\right)^2 - \frac{1}{2}\left(\partial_x\phi\right)^2 - \frac{1}{2}m^2\phi(x)^2 + J\phi \\ \equiv -\frac{1}{2}\left(\nabla\phi\right)^2 - \frac{1}{2}m^2\phi(x)^2 + J\phi,\end{aligned} \tag{3}$$

where I defined $\nabla = (\partial_\tau, \partial_x)$ as the Euclidean 2-divergence. The partition function takes the form

$$\begin{aligned}Z_E[J] &= \int \mathcal{D}\phi\ \exp\left(-\int d^2x\ \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi(x)^2 - J\phi\right) \\ &= \int \mathcal{D}\phi\ \exp\left(-\int d^2x\ \frac{1}{2}\phi\left(-\nabla^2 + m^2\right)\phi - J\phi\right).\end{aligned} \tag{4}$$

I will now integrate out ϕ by writing it as a perturbation about the classical solution, $\phi = \phi_c + \eta$, changing the integrand to

$$\begin{aligned}\frac{1}{2}\phi\left(-\nabla^2 + m^2\right)\phi - J\phi = \frac{1}{2}\phi_c(-\nabla^2 + m^2)\phi_c + \eta(-\nabla^2 + m^2)\phi_c \\ + \frac{1}{2}\eta(-\nabla^2 + m^2)\eta - J\phi_c - J\eta.\end{aligned} \tag{5}$$

The key point is to choose ϕ_c such that the terms linear in η vanish, corresponding to the PDE

$$(-\nabla^2 + m^2)\phi_c = J(x). \tag{6}$$

An expression for ϕ_c is deduced by noting that the Green's function of the operator $-\nabla^2 + m^2$ satisfies

$$(-\nabla^2 + m^2)G_0(x - x') = \delta^2(x - x'), \tag{7}$$

in which case

$$
\begin{aligned}
\phi_c(x) = [-\nabla^2 + m^2]^{-1} J(x) &= \int d^2x' \; [-\nabla^2 + m^2]^{-1} \delta^2(x - x') J(x') \\
&= \int d^2x' \; G_0(x - x') J(x').
\end{aligned} \tag{8}
$$

With this choice, the partition function $Z_E[J]$ is now

$$
\begin{aligned}
Z_E[J] &= e^{-\beta E_0} \int \mathcal{D}\eta \; \exp\left(-\frac{1}{2} \int d^2x \; \eta(-\nabla^2 + m^2)\eta + \frac{1}{2} \iint d^2x d^2x' \; J(x) G_0(x - x') J(x') \right) \\
&= Z_E[0] e^{-\beta E_0} \exp\left(\frac{1}{2} \iint d^2x d^2x' \; J(x) G_0(x - x') J(x') \right),
\end{aligned} \tag{9}
$$

where

$$
Z_E[0] \equiv \int \mathcal{D}\eta \; \exp\left(-\frac{1}{2} \int d^2x \; \eta(-\nabla^2 + m^2)\eta \right) = \left(\det[-\nabla^2 + m^2] \right)^{-1/2} \tag{10}
$$

is the partition function in the absence of sources, which contains information relevant for the ground state energy density.

2 Evaluating the Determinant

I am now in the position to compute the ground state energy, which has the form

$$
\begin{aligned}
E_G = \lim_{\beta\to\infty} \langle E \rangle = -\lim_{\beta\to\infty} \frac{\partial}{\partial\beta} \ln \mathcal{Z}_E[J] &= -\lim_{\beta\to\infty} \frac{\partial}{\partial\beta} \left(\ln \mathcal{Z}_E[0] - \beta E_0 + \frac{1}{2} \iint d^2x d^2x' \; J(x) G_0(x - x') J(x') \right) \\
&= E_0 - \lim_{\beta\to\infty} \frac{\partial}{\partial\beta} \ln \mathcal{Z}_E[0] \\
&= E_0 + \frac{1}{2} \lim_{\beta\to\infty} \frac{\partial}{\partial\beta} \ln\left(\det[-\nabla^2 + m^2] \right).
\end{aligned} \tag{11}
$$

The massless limit, $m \to 0$, is implied in the above expression, and will be taken at the end. In order to evaluate the remaining term, I will analyze the generalized Riemann zeta function for the operator $\hat{M} \equiv -\nabla^2 + m^2$,

$$
\begin{aligned}
\zeta_M(s) = \sum_n \frac{1}{m_n^s} \implies \lim_{s\to 0^+} \frac{d}{ds} \zeta_M(s) = \lim_{s\to 0^+} \sum_n \frac{d}{ds} \left(\frac{1}{m_n^s} \right) = -\lim_{s\to 0+} \sum_n \frac{\ln(m_n)}{m_n^s} &= -\ln\left(\prod_n m_n \right) \\
&= -\ln(\det \hat{M})
\end{aligned} \tag{12}
$$

$$
\iff \ln\left(\det[-\nabla^2 + m^2] \right) = -\lim_{s\to 0^+} \frac{d}{ds} \zeta_M(s), \tag{13}
$$

where $\{m_n\}$ is the set of eigenvalues of the operator $\hat{M}$. To get a nice expression for $\zeta_M(s)$, the trick is to use the Gamma function:

$$
\int_0^\infty dx \; x^{s-1} e^{-m_n x} = \frac{1}{m_n^s} \int_0^\infty dy \; y^{s-1} e^{-y} = \frac{1}{m_n^s} \Gamma(s) \iff \zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty dx \; x^{s-1} \sum_n e^{-m_n x}. \tag{14}
$$

While this is nice, to make progress I need to determine the set of eigenvalues $\{m_n\}$ for the operator $-\nabla^2 + m^2$. I will take a separable ansatz $f_{jk}(x, \tau) = g_j(x) h_k(\tau)$ for the eigenfunctions,

$$
\begin{aligned}
&\left(-\nabla^2 + m^2 \right) f_{jk}(x, \tau) = m_{jk} f_{jk}(x, \tau) \\
\iff &-h_k(\tau) \partial_x^2 g_j(x) - g_j(x) \partial_\tau^2 h_k(\tau) = (m_{jk} - m^2) g_j(x) h_k(\tau) \\
\iff &-\frac{1}{g_j(x)} \partial_x^2 g_j(x) - \frac{1}{h_k(\tau)} \partial_\tau^2 h_k(\tau) \equiv k_j^2 + \omega_k^2
\end{aligned} \tag{15}
$$

where I defined the eigenvalues as $m_{jk} = k_j^2 + \omega_k^2 + m^2$. With periodic boundary conditions in space, $k_j = 2\pi j/L$, and since I am in imaginary time, I have the boundary condition $\omega_k = 2\pi k/\beta$. With these conditions, $\zeta_M(s)$ becomes

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty dx\ x^{s-1} \sum_{j,k} e^{-\left(k_j^2+\omega_k^2+m^2\right)x}. \tag{16}$$

Since I have $\omega_k = 2\pi k/\beta$, at low temperature I can convert the sum over k to an integral using the density of states,

$$\sum_k = \int_{\mathbb{R}} \frac{\beta\, d\omega}{2\pi} \implies \sum_k e^{-\omega^2 x} = \int_{\mathbb{R}} \frac{\beta\, d\omega}{2\pi} e^{-\omega^2 x} = \frac{\beta}{2\pi}\sqrt{\frac{\pi}{x}}, \tag{17}$$

giving

$$\zeta_M(s) = \frac{\beta}{\Gamma(s)} \int_0^\infty \frac{dx}{2\sqrt{\pi}}\ x^{s-3/2} e^{-m^2 x} \sum_j e^{-(2\pi/L)^2 j^2 x}. \tag{18}$$

The trick now is to implement the Poisson summation formula

$$\sum_k f(k) = \sum_l \int_{\mathbb{R}} dy\ e^{2\pi i l y} f(y), \tag{19}$$

resulting in

$$\begin{aligned}\zeta_M(s) = \frac{\beta}{\Gamma(s)} \sum_l \int_0^\infty \frac{dx}{2\sqrt{\pi}}\ x^{s-3/2} e^{-m^2 x} \int_{\mathbb{R}} dy\ e^{2\pi i l y} e^{-(2\pi/L)^2 y^2 x} &= \frac{\beta}{\Gamma(s)} \sum_l \int_0^\infty \frac{dx}{2\sqrt{\pi}}\ x^{s-3/2} e^{-m^2 x} \frac{1}{2\pi}\left(\frac{\pi}{x}\right)^{1/2} e^{-(lL/2)^2/x} \\ &= \frac{\beta L}{4\pi\Gamma(s)} \sum_l \int_0^\infty dx\ x^{s-2} e^{-m^2 x - \frac{l^2L^2}{4x}}.\end{aligned} \tag{20}$$

where I completed the square and evaluated the resulting Gaussian. I now have to be careful in taking the massless limit $m \to 0$. To do this, I will expand the sum over l,

$$\zeta_M(s) = \frac{\beta L}{4\pi\Gamma(s)} \left(\int_0^\infty dx\ x^{s-2} e^{-m^2 x} + 2\sum_{l\geq 1} \int_0^\infty dx\ x^{s-2} e^{-m^2 x - \frac{l^2L^2}{4x}} \right). \tag{21}$$

I can freely let $m \to 0$ in the second integral, but I will leave m in place in the first integral for convergence purposes. This is because the first integral evaluates to

$$\int_0^\infty dx\ x^{s-2} e^{-m^2 x} = \frac{\Gamma(s-1)}{m^{2(s-1)}}, \tag{22}$$

whereas the second is

$$\int_0^\infty dx\ x^{s-2} e^{-\frac{l^2L^2}{4x}} = \left(\frac{lL}{2}\right)^{2(s-1)} \int_0^\infty du\ u^{-s} e^{-u} = \left(\frac{2}{L}\right)^{2(1-s)} \frac{\Gamma(1-s)}{l^{2(1-s)}}, \tag{23}$$

for $m \to 0$. Now, $\zeta_M(s)$ is

$$\begin{aligned}\zeta_M(s) &= \frac{\beta L}{4\pi\Gamma(s)} \left(\frac{\Gamma(s-1)}{m^{2(s-1)}} + 2\left(\frac{2}{L}\right)^{2(1-s)} \Gamma(1-s) \sum_{l\geq 1} \frac{1}{l^{2(1-s)}} \right) \\ &= \frac{\beta L}{4\pi} \left(\frac{1}{m^{2(s-1)}(s-1)} + \frac{2}{\pi}\left(\frac{2}{L}\right)^{2(1-s)} \zeta(2(1-s))\Gamma^2(1-s)\sin(\pi s) \right),\end{aligned} \tag{24}$$

where I made use of the gamma function identities

$$\frac{\Gamma(s-1)}{\Gamma(s)} = \frac{1}{s-1}, \qquad\qquad \Gamma(1-s)\Gamma(s) = \pi\csc(\pi s). \tag{25}$$

Now that I have finally determined $\zeta_M(s)$, I can use it to evaluate $\ln\det[-\nabla^2 + m^2]$. From (13), this is

$$\begin{aligned}\ln(\det[-\nabla^2+m^2]) = -\frac{\beta L}{4\pi}\Big(\lim_{s\to 0^+}\frac{d}{ds}\left[\frac{1}{m^{2(s-1)}(s-1)}\right] \\ +\frac{2}{\pi}\left(\frac{2}{L}\right)^2\lim_{s\to 0^+}\frac{d}{ds}\left[\left(\frac{2}{L}\right)^{-2s}\zeta(2(1-s))\Gamma^2(1-s)\sin(\pi s)\right]\Big).\end{aligned} \tag{26}$$

The first term is easy since I can first expand to first order in s, then differentiate and set $s\to 0^+$,

$$\begin{aligned}\frac{1}{m^{2(s-1)}(s-1)} &= -m^2 m^{-2s}\cdot\frac{1}{1-s} = -m^2 e^{-2s\ln(m/\mu)}\left(1+s+\mathcal{O}(s^2)\right) \\ &= -m^2\left(1+s(1-2\ln(m/\mu))+\mathcal{O}(s^2)\right) \\ \iff \lim_{s\to 0^+}\frac{d}{ds}\left[\frac{1}{m^{2(s-1)}(s-1)}\right] &= m^2(2\ln(m/\mu)-1),\end{aligned} \tag{27}$$

where I introduced a temporary dimensional cutoff μ to perform the expansion. Using Mathematica for the second term, I find

$$\begin{aligned}&\frac{d}{ds}\left[\left(\frac{2}{L}\right)^{-2s}\zeta(2(1-s))\Gamma^2(1-s)\sin(\pi s)\right] \\ &= \left(\frac{2}{L}\right)^{-2s}\Gamma^2(1-s)\Big(\zeta(2(1-s))\left(2\sin(\pi s)\ln(L/2)+\pi\cos(\pi s)-2\sin(\pi s)\psi^{(0)}(1-s)\right) \\ &\qquad -2\sin(\pi s)\frac{d}{ds}\zeta(2(1-s))\Big),\end{aligned} \tag{28}$$

where $\psi^{(0)}$ is the digamma function, or logarithmic derivative of the gamma function. Since $\Gamma(1)=1$, $\zeta(2)=\pi^2/6$, $\psi^{(0)}(1)=-\gamma$, and $\frac{d}{ds}\zeta(2(1-s))$ is finite as $s\to 0^+$, the result is simple,

$$\lim_{s\to 0^+}\frac{d}{ds}\left[\left(\frac{2}{L}\right)^{-2s}\zeta(2(1-s))\Gamma^2(1-s)\sin(\pi s)\right] = \frac{\pi^3}{6}, \tag{29}$$

such that (26) can be written as

$$\ln(\det[-\nabla^2+m^2]) = -\frac{\beta L}{4\pi}\lim_{m\to 0}\left(m^2(2\ln(m/\mu)-1)+\frac{4\pi^2}{3L^2}\right) = -\frac{\pi}{3L}\beta \tag{30}$$

in the massless limit. At last, (11) tells me the shift in the ground state energy is

$$E_{\text{fluc.}} \equiv E_G - E_0 = \frac{1}{2}\lim_{\beta\to\infty}\frac{\partial}{\partial\beta}\left(-\frac{\pi}{3L}\beta\right) = -\frac{\pi}{6L}, \tag{31}$$

which vanishes with increasing system length. In $1+1$ dimensions, force has the dimensions of energy per length. Inside of the system there is a fluctuation energy $E_{\text{fluc.}}$, whereas outside the system I have a divergent energy density since the field momenta is not quantized. If I interpret $E_{\text{fluc.}}$ as the energy inside the system and the higher-order terms I neglected in E_G as the energy outside the system, there is an effective attractive force on the walls of the system

$$\boxed{F_{\text{Casimir}} = \frac{E_{\text{in}}-E_{\text{out}}}{L} = -\frac{\pi}{6L^2}} \tag{32}$$

by the zero-point motion of the field, despite this being in vacuum.

References

[1] Eduardo Fradkin. *Quantum Field Theory: An Integrated Approach.* Princeton University Press, Princeton and Oxford, 2021.