Mapping the Anisotropic non-linear Sigma Model to the sine-Gordon Model

Ethan Huecker

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1 Derivation

Our focus is the anisotropic nLSM of the form

$$S[\phi] = \int_x \left[\frac{\kappa_1}{2}(\partial_x \phi)^2 + \frac{\kappa_2}{2}(\partial_y \phi)^2\right], \tag{1}$$

and we wish to derive the associated sine-Gordon model, which lies in the same universality class. To perform this mapping, we first note that this action describes the continuum limit of an XY model with partition function

$$\mathcal{Z} = \int \mathcal{D}\phi \exp\left(\kappa_1 \sum_r \cos(\Delta_x \phi(r)) + \kappa_2 \sum_r \cos(\Delta_y \phi(r))\right), \tag{2}$$

which can be rewritten in the form (see (6))

$$\mathcal{Z} = \prod_{r,\mu} \sum_{n_\mu(r)=-\infty}^{\infty} \int \mathcal{D}\phi \exp\left(-\frac{n_\mu^2(r)}{2\kappa_\mu} + i n_\mu(r) \Delta_\mu \phi(r)\right), \tag{3}$$

where $\int \mathcal{D}\phi = \prod_r \int_{-\pi}^{\pi} d\phi(r)$.

Proof: This will rely heavily on the Kogut paper [1]. For $\mu = x, y$, we can Fourier transform the bond variables,[a]

$$\exp(-\kappa_\mu(1 - \cos(\Delta_\mu \phi(r)))) = \sum_{n_\mu(r)=-\infty}^{\infty} \exp(i n_\mu(r) \Delta_\mu \phi(r)) I_{n_\mu(r)}(\kappa_\mu), \tag{4}$$

where Δ_μ is the finite difference operator in direction μ. Noting the asymptotic behavior of the modified Bessel function,

$$I_{n_\mu(r)}(\kappa_\mu) = \frac{e^{\kappa_\mu}}{\sqrt{2\pi\kappa_\mu}}\left(1 - \frac{4n_\mu^2(r) - 1}{8\kappa_\mu} + \mathcal{O}(\kappa_\mu^{-2})\right) \approx \frac{e^{\kappa_\mu}}{\sqrt{2\pi\kappa_\mu}} \exp\left(-\frac{4n_\mu^2(r) - 1}{8\kappa_\mu}\right) \approx C \exp\left(-\frac{n_\mu^2(r)}{2\kappa_\mu}\right), \tag{5}$$

where C is a constant which can pulled outside the partition function, I can equivalently write

$$\exp(\kappa_\mu \cos(\Delta_\mu \phi(r))) \propto \sum_{n_\mu(r)=-\infty}^{\infty} \exp\left(i n_\mu(r) \Delta_\mu \phi(r) - \frac{n_\mu^2(r)}{2\kappa_\mu}\right). \tag{6}$$

[a] This form can also be reached by first using the Villain transformation, then exploiting the Poisson summation formula.

At this point the integration over ϕ is trivial, as ϕ can be viewed as a Lagrange multiplier ensuring the constraint $\Delta_\mu n_\mu(r) = 0$. Hence, $n_\mu(r)$ is divergence-less and can be written as the curl of a integer valued scalar field $n(r)$: $n_\mu(r) = \varepsilon_{\mu\nu} \Delta_\nu n(r)$, where $\varepsilon_{\mu\nu}$ is the 2d Levi-Civita symbol. We have

$$\mathcal{Z} = \prod_{r,\mu} \sum_{n_\mu(r)=-\infty}^{\infty} \delta_{\Delta_\mu n_\mu(r),0} \exp\left(-\frac{n_\mu^2(r)}{2\kappa_\mu}\right) = \sum_{n(r)=-\infty}^{\infty} \exp\left(-\frac{1}{2\kappa_1}\sum_r (\Delta_x n(r))^2 - \frac{1}{2\kappa_2}\sum_r (\Delta_y n(r))^2\right), \tag{7}$$

where I used the fact that $n_\mu^2 = \varepsilon_{\mu\nu}\varepsilon_{\mu\rho}\Delta_\nu n(r)\Delta_\rho n(r) = (\Delta_\mu n(r))^2$. Now, we use the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\tilde{\phi}\, g(\tilde{\phi}) \exp(2\pi i m\tilde{\phi}), \tag{8}$$

noting that here $n = n(r)$, giving

$$\mathcal{Z} = \left[\prod_r \int_{-\infty}^{\infty} d\tilde{\phi}(r)\right] \sum_{m(r)=-\infty}^{\infty} \exp\left(-\frac{1}{2\kappa_1}\sum_r (\Delta_x\tilde{\phi}(r))^2 - \frac{1}{2\kappa_2}\sum_r (\Delta_y\tilde{\phi}(r))^2 + 2\pi i \sum_r m(r)\tilde{\phi}(r)\right). \tag{9}$$

In deriving the sine-Gordon theory we use some key insight [1]. As high-frequency modes are integrated out, chemical potential and long-range logarithmic interaction terms are generated (see Sec 2 for more details). Anticipating the generation of a chemical potential term $\ln y \sum_r m^2(r)$ through RG, we rewrite (9) as

$$\mathcal{Z} = \int \mathcal{D}\phi \exp\left(-\frac{1}{2\kappa_1}\sum_r (\Delta_x\tilde{\phi}(r))^2 - \frac{1}{2\kappa_2}\sum_r (\Delta_y\tilde{\phi}(r))^2\right) \sum_{m(r)=-\infty}^{\infty} \exp\left(\ln y \sum_r m^2(r) + 2\pi i \sum_r m(r)\tilde{\phi}(r)\right). \tag{10}$$

For small fugacities, it is sufficient to keep the terms $m(r) = -1, 0, 1$ in the summation over $m(r)$, as $|\ln y| \gg 1$. It follows that

$$\begin{aligned}
\sum_{m(r)=0,\pm1} \exp\left(\ln y \sum_r m^2(r) + 2\pi i \sum_r m(r)\tilde{\phi}(r)\right) &= \prod_r \sum_{m(r)=0,\pm1} \exp\left(\ln y\, m^2(r) + 2\pi i m(r)\tilde{\phi}(r)\right) \\
&= \prod_r \left(1 + 2y\cos(2\pi\tilde{\phi}(r))\right) \\
&\approx \exp\left(2y \sum_r \cos(2\pi\tilde{\phi}(r))\right),
\end{aligned} \tag{11}$$

and so (10) becomes

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}\phi \exp\left(-\sum_r \left(\frac{1}{2\kappa_1}(\Delta_x\tilde{\phi}(r))^2 + \frac{1}{2\kappa_2}(\Delta_y\tilde{\phi}(r))^2 - 2y\cos(2\pi\tilde{\phi}(r))\right)\right) \\
&= \int \mathcal{D}\phi \exp\left(-\sum_r \left(\frac{1}{2\sqrt{\kappa_1\kappa_2}}\sqrt{\frac{\kappa_2}{\kappa_1}}(\Delta_x\tilde{\phi}(r))^2 + \frac{1}{2\sqrt{\kappa_1\kappa_2}}\sqrt{\frac{\kappa_1}{\kappa_2}}(\Delta_y\tilde{\phi}(r))^2 - 2y\cos(2\pi\tilde{\phi}(r))\right)\right) \\
&\equiv \int \mathcal{D}\phi \exp\left(-\sum_r \left(\frac{1}{2\sqrt{\kappa_1\kappa_2}}|\mathbf{D}'\tilde{\phi}(r)|^2 - g_\phi\cos(2\pi\tilde{\phi}(r))\right)\right),
\end{aligned} \tag{12}$$

which is the dual sine-Gordon theory. As can be seen, the stiffnesses are simply inverted; I have defined the derivative

$$\mathbf{D}' = \sqrt{\frac{\kappa_2}{\kappa_1}}\hat{x}\partial_x + \sqrt{\frac{\kappa_1}{\kappa_2}}\hat{y}\partial_y \tag{13}$$

to contrast with the original nLSM

$$S[\phi] = \int_x \left[\frac{\sqrt{\kappa_1\kappa_2}}{2}|\mathbf{D}\phi|^2\right] \quad \text{where} \quad \mathbf{D} = \sqrt{\frac{\kappa_1}{\kappa_2}}\hat{x}\partial_x + \sqrt{\frac{\kappa_2}{\kappa_1}}\hat{y}\partial_y. \tag{14}$$

2 Aside On Coulomb-Gas Representation

I will investigate the Coulomb gas representation, in which $\tilde{\phi}$ has been completely integrated out, to justify the introduction of a chemical potential term (10). The integration over $\tilde{\phi}$ in (9) is a standard Gaussian, resulting in

$$\mathcal{Z} = \sum_{m(r)=-\infty}^{\infty} \exp\left(-2\pi^2 \sum_{r,r'} m(r) G(r-r') m(r')\right), \tag{15}$$

where $G(r-r')$, for large $|r-r'|$, can be approximated by its continuum representation

$$G(r) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i(k_x x + k_y y)}}{k_x^2/\kappa_1 + k_y^2/\kappa_2} = \sqrt{\kappa_1\kappa_2} \int \frac{d^2k}{(2\pi)^2} \frac{e^{i(k_x(\sqrt{\kappa_1}x) + k_y(\sqrt{\kappa_2}y))}}{k^2}. \tag{16}$$

It is clear that $G(0)$ is divergent. Isolating this divergence,

$$\mathcal{Z} = \sum_{m(r)} \exp\left(-2\pi^2 G(0)\left(\sum_r m(r)\right)^2 - 2\pi^2 \sum_{r,r'} m(r) G'(r-r') m(r')\right), \tag{17}$$

it is clear to see that the divergence of $G(0)$ singles out neutral configurations $\sum_r m(r) = 0$. Hence,

$$\mathcal{Z} = \sum_{m(r)}{}' \exp\left(-2\pi^2 \sum_{r,r'} m(r) G'(r-r') m(r')\right), \tag{18}$$

where the prime on the sum over $m(r)$ signifies a sum over neutral configurations only. The convergent part $G'(r-r')$ can be approximated by [1]

$$G'(r-r') \approx -\frac{\sqrt{\kappa_1\kappa_2}}{2\pi} \ln \sqrt{\kappa_1(x-x')^2 + \kappa_2(y-y')^2} - \frac{\sqrt{\kappa_1\kappa_2}}{4}, \tag{19}$$

from its structure in (16). Hence,

$$\mathcal{Z} = \sum_{m(r)}{}' \exp\left(\frac{\pi^2\sqrt{\kappa_1\kappa_2}}{2} \sum_{r\neq r'} m(r)m(r') + \pi\sqrt{\kappa_1\kappa_2} \sum_{r\neq r'} m(r) \ln \sqrt{\kappa_1(x-x')^2 + \kappa_2(y-y')^2}\, m(r')\right), \tag{20}$$

which can be further simplified by imposing the neutrality condition $\sum_r m(r) = 0$, such that

$$0 = \sum_{r,r'} m(r)m(r') = \sum_{r\neq r'} m(r)m(r') + \sum_r m^2(r), \tag{21}$$

giving

$$\mathcal{Z} = \sum_{m(r)}{}' \exp\left(-\frac{\pi^2\sqrt{\kappa_1\kappa_2}}{2} \sum_r m^2(r) + \pi\sqrt{\kappa_1\kappa_2} \sum_{r\neq r'} m(r) \ln \sqrt{\kappa_1(x-x')^2 + \kappa_2(y-y')^2}\, m(r')\right). \tag{22}$$

Therefore, we see a chemical potential term with $\ln y = -\pi^2\sqrt{\kappa_1\kappa_2}/2$, and an anisotropic logarithmic interaction term with interaction strength $\sqrt{\kappa_1\kappa_2}$. This interaction strength, $\sqrt{\kappa_1\kappa_2}$, motivates absorbing the anisotropy into derivatives.

3 References

[1] J. Kogut. An introduction to lattice gauge theory and spin systems. Rev. Mod. Phys. 51, 659.