Coadjoint Orbit Bosonization of the Fermi Surface and Magnetic Response

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1 Lie Algebra

Lie Algebra: A vector space $\mathfrak{g}$ equipped with a binary operation known as the Lie bracket

$$[\cdot,\cdot]: \ \mathfrak{g} \mapsto \mathfrak{g} \tag{1}$$

such that $\forall X, Y, Z \in \mathfrak{g}$ and $a, b \in \mathbb{R}$ or $\mathbb{C}$

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \qquad [X, Y] = -[Y, X], \qquad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \tag{2}$$

the latter of which being the Jacobi identity. Given a set of generators (basis) for the Lie algebra, $\{\lambda_i\}$, typically normalized by $\mathrm{tr}[\lambda_a \lambda_b] = \delta_{ab}/2$, the Lie bracket between them is

$$[\lambda_a, \lambda_b] = f_{ab}^{\ \ c} \lambda_c, \tag{3}$$

where $f_{ab}^{\ \ c}$ are the structure constants of the Lie algebra.

The algebra is an abstract concept, and is more grounded through representations in the form of matrices. It is often the case that these representations act on vector spaces external to $\mathfrak{g}$; the *adjoint* representation, on the other hand, is a representation which furnishes the intrinsic structure of $\mathfrak{g}$:

$$\mathrm{ad}_X: \ \mathfrak{g} \mapsto \mathfrak{g}, \qquad \mathrm{ad}_X Y = [X, Y] \tag{4}$$

Hence, $\mathrm{ad}: \ \mathfrak{g} \mapsto \mathfrak{gl}(\mathfrak{g})$, where $\mathfrak{gl}(\mathfrak{g})$ is the set of all linear maps from $\mathfrak{g}$ to itself. This can be seen from the Jacobi identity,

$$\begin{aligned} 0 &= [\lambda_i, [\lambda_j, \lambda_k]] + [\lambda_j, [\lambda_k, \lambda_i]] + [\lambda_k, [\lambda_i, \lambda_j]] \\ \iff [[\lambda_j, \lambda_k], \lambda_i] &= [\lambda_j, [\lambda_k, \lambda_i]] - [\lambda_k, [\lambda_j, \lambda_i]] \\ \iff \mathrm{ad}_{[\lambda_j, \lambda_k]} \lambda_i &= [\lambda_j, \mathrm{ad}_{\lambda_k} \lambda_i] - [\lambda_k, \mathrm{ad}_{\lambda_j} \lambda_i] \\ &= \mathrm{ad}_{\lambda_j} \mathrm{ad}_{\lambda_k} \lambda_i - \mathrm{ad}_{\lambda_k} \mathrm{ad}_{\lambda_j} \lambda_i, \end{aligned} \tag{5}$$

such that

$$[\mathrm{ad}_{\lambda_j}, \mathrm{ad}_{\lambda_k}] = \mathrm{ad}_{[\lambda_j, \lambda_k]} = f_{jk}^{\ \ i} \, \mathrm{ad}_{\lambda_i} \tag{6}$$

which is nothing but (3), albeit in terms of a representation. Clearly, the adjoint representation is the identification of the generators of the algebra with the associated structure constants.

Lie Group: A group[1] G that is also a smooth manifold, i.e. has a continuous, differentiable structure. The group operations are necessarily smooth maps, allowing for the description of infinitesimal transformations. The local structure of a Lie group is encoded by its associated Lie algebra, which is simply the tangent space at the identity. As such, G is generated from $\mathfrak{g}$ by the exponential map, $\exp: \ \mathfrak{g} \mapsto G$.

The structure constants encode the local structure of the Lie group, as the Baker-Campbell-Hausdorff (BCS) formula

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \cdots\right), \tag{7}$$

which represents infinitesimal group multiplication near the identity, depends explicitly on $f_{ab}^{\ \ c}$ through the commutator. By taking an exponential map of the adjoint action (4), we deduce the Lie group adjoint action

$$\mathrm{Ad}_{\exp X}: \ \mathfrak{g} \mapsto \mathfrak{g} \qquad \mathrm{Ad}_{\exp X} = e^{\mathrm{ad}_X}. \tag{8}$$

[1] Closed set with a binary operation satisfying associativity, identity, and inverses

Explicitly, on members $Y \in \mathfrak{g}$

$$\mathrm{Ad}_{\exp X} Y = e^{\mathrm{ad}_X} Y = e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots . \tag{9}$$

By defining group elements through $U = \exp X \in G$, the connection with unitary transformations in quantum mechanics is evident.

Example: $\mathfrak{su}(2)$
The Lie algebra $\mathfrak{su}(2)$ is the set of 2×2 traceless Hermitian complex matrices known as the Pauli matrices $\sigma_i = (\sigma_1, \sigma_2, \sigma_3)$. Obeying the normalization convention, the generators are half of the Pauli matrices. Any element of $\mathfrak{su}(2)$ can be written as $X = i n_i \sigma_i$, where $n_i \in \mathbb{R}^3$. Every element of the Lie group SU(2) can be written as an exponential of the algebra

$$U = \exp(i\theta \hat{n}_i \sigma_i) = \cos(\theta) I + i \sin(\theta) n_i \sigma_i, \tag{10}$$

where $||\hat{n}_i|| = 1$ and $\theta \in \mathbb{R}$. The Pauli matrices obey

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k, \tag{11}$$

so the structure constants are $f_{ijk} = i\varepsilon_{ijk}$ [5, 6].

Dual Space: The dual space $\mathfrak{g}^*$ to a Lie algebra $\mathfrak{g}$ is the set of linear functionals on $\mathfrak{g}$, $\{\mu\}$, such that

$$\mu : \ X \in \mathfrak{g} \mapsto \langle \mu, X \rangle \in \mathbb{R}. \tag{12}$$

This allows us to define an associated *coadjoint action* on elements of the dual space g^*, ad^*_X, such that $\langle \mu, \mathrm{ad}_X Y \rangle = -\langle \mathrm{ad}^*_X \mu, Y \rangle$ for $\mu \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$.[2] Explicitly,

$$\mathrm{ad}^*_X : \ \mathfrak{g}^* \mapsto \mathfrak{g}^*, \qquad \mathrm{ad}^*_X \mu = [X, \mu]. \tag{13}$$

This generates a group coadjoint action

$$\mathrm{Ad}^*_{\exp X} : \ \mathfrak{g}^* \mapsto \mathfrak{g}^* \qquad \mathrm{Ad}^*_{\exp X} = e^{\mathrm{ad}^*_X}, \tag{14}$$

with similar properties to the group adjoint action, except now $\mu \in \mathfrak{g}^*$:

$$\mathrm{Ad}^*_{\exp X} \mu = e^{\mathrm{ad}^*_X} \mu = e^X \mu e^{-X} = \mu + [X,\mu] + \frac{1}{2!}[X,[X,\mu]] + \frac{1}{3!}[X,[X,[X,\mu]]] + \cdots . \tag{15}$$

Lie Algebra of Canonical Transformations

Good references for this section are [3, 8]. Define a $2d$ phase space of canonical coordinates, $(\mathbf{x}, \mathbf{p})$. General elements of the Lie algebra of canonical transformations $\mathfrak{g}$ are phase space functions $F(\mathbf{x}, \mathbf{p})$, to be identified as observables (Hamiltonian, angular momentum, etc.). Elements of the dual space $\mathfrak{g}^*$ should be that which when associated with an observable result in a physical quantity (like an element of $\mathbb{R}$, say). Naturally then, elements of $\mathfrak{g}^*$ are phase space distribution functions $f(\mathbf{x}, \mathbf{p})$ with an associated scalar product

$$\langle f, F \rangle \equiv \int \frac{d^d x d^d p}{(2\pi)^d} F(\mathbf{x}, \mathbf{p}) f(\mathbf{x}, \mathbf{p}), \tag{16}$$

interpretted as the expectation value of the observable F in state f. The Poisson bracket is given by

$$\{F, G\} = \nabla_{\mathbf{x}} F \cdot \nabla_{\mathbf{p}} G - \nabla_{\mathbf{p}} F \cdot \nabla_{\mathbf{x}} G, \tag{17}$$

which I define to be the Lie bracket (adjoint action) on $\mathfrak{g}$. Observables are taken to be elements of $\mathfrak{g}$ since they generate dynamics through Hamiltonian flow,

$$X_F = \frac{\partial F}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial}{\partial p_i} = -\{F, \ \ \} \tag{18}$$

[2] This sign ensures that $\langle \mu, \mathrm{Ad}_U Y \rangle = \langle \mathrm{Ad}^*_U \mu, Y \rangle$ for group action. This is in the same spirit as ensuring the trace operation has the cyclic property $\mathrm{tr}[F e^U G e^{-U}] = \mathrm{tr}[e^{-U} F e^U G]$.

where X_F is the Hamiltonian vector field for the observable F (not necessarily the Hamiltonian). For a Hamiltonian H, we have

$$\dot{x}_i = X_H x_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = X_H p_i = -\frac{\partial H}{\partial x_i}, \tag{19}$$

which are just Hamilton's equations. It is evident from this construction that $X_F f = -\operatorname{ad}^*_F f$; as example, take a Hamiltonian $H = \varepsilon(\mathbf{p}) + V(\mathbf{x}) \in \mathfrak{g}$, where since

$$\operatorname{ad}^*_H f = \{H, f\} = \nabla_{\mathbf{x}} V \cdot \nabla_{\mathbf{p}} f - \nabla_{\mathbf{p}} \varepsilon(\mathbf{p}) \cdot \nabla_{\mathbf{x}} f, \tag{20}$$

then

$$\begin{aligned} \frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \nabla_{\mathbf{x}} f + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f &= \frac{\partial f}{\partial t} + \nabla_{\mathbf{p}} \varepsilon(\mathbf{p}) \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} V \cdot \nabla_{\mathbf{p}} f \\ &= \frac{\partial f}{\partial t} - \operatorname{ad}^*_H f, \end{aligned} \tag{21}$$

which is the Boltzmann equation! This appearance makes sense, the Hamiltonian generates time evolution through $X_H f = -\operatorname{ad}^*_H f$.

Aside: The Boltzmann equation (BE) is useful in the determination of transport/response properties with its incorporation of collisions between particles. This is done with a collision integral $(\partial_t f)_{\text{coll}}$, such that

$$\partial_t f + \nabla_{\mathbf{p}} \varepsilon(\mathbf{p}) \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} V \cdot \nabla_{\mathbf{p}} f = (\partial_t f)_{\text{coll}}. \tag{22}$$

in the absence of collisions, $(\partial_t f)_{\text{coll}=0}$, this is the Liouville equation. In solid-state physics, it is common to use the relaxation-time approximation $(\partial_t f)_{\text{coll}} \approx -(f - f_0)/\tau$. Here, τ is the relaxation-time, or characteristic timescale for f to relax to the equilibrium distribution f_0 [2]. Notice here how the equilibrium distribution is the input; the Fermi-Dirac or Bose-Einstein distributions are not "derived" from this equation. The BE plays a key role in semi-classical transport theory, and use to derive responses like conductivity σ (through Kubo formula), thermo-electric coefficients, etc. The BE makes use in Fermi liquid theory as the equation describing the evolution of the *quasi-particle* distribution function $f(\mathbf{x}, \mathbf{p}, t)$ in phase space.

In the absence of collisions, $df/dt = 0$, and we arrive at the Liouville equation

$$\partial_t f = \operatorname{ad}^*_H f. \tag{23}$$

All of these results are valid in the semi-classical picture. Properly quantized dynamics are described by the Moyal algebra, which limit to the Poisson algebra when $\hbar \to 0$.

2 Moyal Algebra

2.1 Second-Quantized Approach

I will first show how the Moyal algebra arises from the algebra of fermionic bilinears. This is the second-quantized approach introduced in [8], providing a connection to microscopic fermions and more natural for the development of the theory. The reason for working with such a generator is that it makes clear the generation of particle-hole excitations to the FS and how the FS deforms. It is often convenient to work in c.o.m coordinates, defining

$$T(\mathbf{x},\mathbf{y}) = \frac{i}{2}\left(\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right) \tag{24}$$

for the purposes of realizing the algebra. In anticipation of working on a phase space, I introduce the Wigner transformation of the generators

$$T(\mathbf{x},\mathbf{p}) = \int d^dy\, T(\mathbf{x},\mathbf{y})e^{i\mathbf{p}\cdot\mathbf{y}}, \qquad T(\mathbf{q},\mathbf{y}) = \int d^dx\, T(\mathbf{x},\mathbf{y})e^{i\mathbf{q}\cdot\mathbf{x}}. \tag{25}$$

Note that $T(\mathbf{q},\mathbf{y})$ is the Fourier conjugate to $T(\mathbf{x},\mathbf{p})$ on the base (phase) space, where I have defined Fourier transforms through

$$T(\mathbf{x},\mathbf{p}) = \int \frac{d^dq d^dy}{(2\pi)^d}\, T(\mathbf{q},\mathbf{y})e^{-i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{y}}. \tag{26}$$

Basically, the base phase space is accessed by doing a Wigner transform over the relative coordinate, while the conjugate phase space is accessed by doing so over the c.o.m coordinate. Note that $T^\dagger(\mathbf{x},\mathbf{p}) = -T(\mathbf{x},\mathbf{p})$ is anti-Hermitian.

How do I know that these generators form an algebra, and what are the structure constants of such algebra? The first step to showing this is by proving

$$[T(\mathbf{q},\mathbf{y}),T(\mathbf{q}',\mathbf{y}')] = 2\sin\left(\frac{\mathbf{q}'\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{y}'}{2}\right)T(\mathbf{q}+\mathbf{q}',\mathbf{y}+\mathbf{y}'), \tag{27}$$

i.e. that $T(\mathbf{q},\mathbf{y})$ is closed. This is shown in detail below.

Proof of Closure of Fermion Bilinears (27): Here,

$$T(\mathbf{q},\mathbf{y}) = \int d^dx\, T(\mathbf{x},\mathbf{y})e^{i\mathbf{q}\cdot\mathbf{x}} = \frac{i}{2}\int d^dx\left(\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right)e^{i\mathbf{q}\cdot\mathbf{x}}, \tag{28}$$

which allows me to write the commutator as

$$\begin{aligned}[T(\mathbf{q},\mathbf{y}),T(\mathbf{q}',\mathbf{y}')] = -\frac{1}{4}\int d^dxd^dx'\Big[&\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right),\\ &\psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)-\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)\psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\Big]e^{i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{q}'\cdot\mathbf{x}'}.\end{aligned} \tag{29}$$

It is easy to show for anti-commuting fermions that $[\psi_i^\dagger\psi_j,\psi_k^\dagger\psi_l] = \delta_{j,k}\psi_i^\dagger\psi_l-\delta_{i,l}\psi_k^\dagger\psi_j$. It follows that

$$\begin{aligned}\left[\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right),\psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)\right] &= \delta_{\mathbf{x}-\mathbf{y}/2,\mathbf{x}'+\mathbf{y}'/2}\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)\\ &-\delta_{\mathbf{x}+\mathbf{y}/2,\mathbf{x}'-\mathbf{y}'/2}\psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right).\end{aligned} \tag{30}$$

The commutator in (29) expands to four commutators, each of which are identical (30) by anti-commuting fermions. Hence,

$$
\begin{aligned}
[T(\mathbf{q},\mathbf{y}),T(\mathbf{q}',\mathbf{y}')] &= -\int d^dx d^dx' \Big(\delta_{\mathbf{x}-\mathbf{y/2},\mathbf{x}'+\mathbf{y}'/2}\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right) \\
&\qquad\qquad - \delta_{\mathbf{x}+\mathbf{y/2},\mathbf{x}'-\mathbf{y}'/2}\psi^\dagger\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\Big)e^{i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{q}'\cdot\mathbf{x}'} \\
&= -\int d^dx\Big(e^{-i\mathbf{q}'\cdot(\mathbf{y}+\mathbf{y}')/2}\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}-\mathbf{y}'\right) \\
&\qquad\qquad - e^{i\mathbf{q}'\cdot(\mathbf{y}+\mathbf{y}')/2}\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}}{2}+\mathbf{y}'\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\Big)e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}.
\end{aligned} \tag{31}
$$

In the first term I shift $\mathbf{x}\to\mathbf{x}+\mathbf{y}'/2$, while in the second term I shift $\mathbf{x}\to\mathbf{x}-\mathbf{y}'/2$. The bilinear is then the same in both, and can be pulled out:

$$
\begin{aligned}
[T(\mathbf{q},\mathbf{y}),T(\mathbf{q}',\mathbf{y}')] &= -\int d^dx\Big(e^{i(\mathbf{q}\cdot\mathbf{y}'-\mathbf{q}'\cdot\mathbf{y})/2}-e^{-i(\mathbf{q}\cdot\mathbf{y}'-\mathbf{q}'\cdot\mathbf{y})/2}\Big)\psi^\dagger\left(\mathbf{x}+\frac{\mathbf{y}+\mathbf{y}'}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}\right)e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}} \\
&= -2i\sin\left(\frac{\mathbf{q}\cdot\mathbf{y}'-\mathbf{q}'\cdot\mathbf{y}}{2}\right)(-iT(\mathbf{q}+\mathbf{q}',\mathbf{y}+\mathbf{y}')) \\
&= 2\sin\left(\frac{\mathbf{q}'\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{y}'}{2}\right)T(\mathbf{q}+\mathbf{q}',\mathbf{y}+\mathbf{y}'),
\end{aligned} \tag{32}
$$

where I note the Wigner transformation (25).

The commutator relation (27) can be Fourier transformed back to the proper basis of generators $T(\mathbf{x},\mathbf{p})$,

$$
\begin{aligned}
&\int \frac{d^dxd^dp}{(2\pi)^d}\frac{d^dx'd^dp'}{(2\pi)^d}[T(\mathbf{x},\mathbf{p}),T(\mathbf{x}',\mathbf{p}')]e^{i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{q}'\cdot\mathbf{x}'}e^{-i\mathbf{p}\cdot\mathbf{y}}e^{-i\mathbf{p}'\cdot\mathbf{y}'} \\
&\qquad = 2\sin\left(\frac{\mathbf{q}'\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{y}'}{2}\right)\int\frac{d^dxd^dp}{(2\pi)^d}T(\mathbf{x},\mathbf{p})e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}e^{-i(\mathbf{y}+\mathbf{y}')\cdot\mathbf{p}} \\
&\qquad = (2\pi)^d\int\frac{d^dxd^dp}{(2\pi)^d}\frac{d^dx'd^dp'}{(2\pi)^d}\left[\sin\left(\frac{\mathbf{q}'\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{y}'}{2}\right)e^{i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{q}'\cdot\mathbf{x}'}e^{-i\mathbf{p}\cdot\mathbf{y}}e^{-i\mathbf{p}'\cdot\mathbf{y}'}\right] \\
&\qquad\qquad \times\left[2\delta_{\mathbf{x},\mathbf{x}'}\delta_{\mathbf{p},\mathbf{p}'}T(\mathbf{x},\mathbf{p})\right].
\end{aligned} \tag{33}
$$

The sine can be viewed as a differential operator acting on the four exponentials by making the replacements $\mathbf{q}\to -i\nabla_{\mathbf{x}}$, $\mathbf{q}'\to -i\nabla_{\mathbf{x}'}$, $\mathbf{y}\to i\nabla_{\mathbf{p}}$, and $\mathbf{y}'\to i\nabla_{\mathbf{p}'}$,

$$
\begin{aligned}
&\int \frac{d^dxd^dp}{(2\pi)^d}\frac{d^dx'd^dp'}{(2\pi)^d}[T(\mathbf{x},\mathbf{p}),T(\mathbf{x}',\mathbf{p}')]e^{i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{q}'\cdot\mathbf{x}'}e^{-i\mathbf{p}\cdot\mathbf{y}}e^{-i\mathbf{p}'\cdot\mathbf{y}'} \\
&= (2\pi)^d\int\frac{d^dxd^dp}{(2\pi)^d}\frac{d^dx'd^dp'}{(2\pi)^d}\left[\sin\left(\frac{\nabla_{\mathbf{x}'}\cdot\nabla_{\mathbf{p}}-\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{p}'}}{2}\right)e^{i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{q}'\cdot\mathbf{x}'}e^{-i\mathbf{p}\cdot\mathbf{y}}e^{-i\mathbf{p}'\cdot\mathbf{y}'}\right]\times\left[2\delta_{\mathbf{x},\mathbf{x}'}\delta_{\mathbf{p},\mathbf{p}'}T(\mathbf{x},\mathbf{p})\right] \\
&= \int\frac{d^dxd^dp}{(2\pi)^d}\frac{d^dx'd^dp'}{(2\pi)^d}e^{i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{q}'\cdot\mathbf{x}'}e^{-i\mathbf{p}\cdot\mathbf{y}}e^{-i\mathbf{p}'\cdot\mathbf{y}'}\times 2(2\pi)^d\sin\left(\frac{\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{p}'}-\nabla_{\mathbf{x}'}\cdot\nabla_{\mathbf{p}}}{2}\right)\left[\delta_{\mathbf{x},\mathbf{x}'}\delta_{\mathbf{p},\mathbf{p}'}T(\mathbf{x},\mathbf{p})\right]
\end{aligned} \tag{34}
$$

where I noted that the sine operator is self-adjoint in the second equality. Matching the integrands, I find

$$
[T(\mathbf{x},\mathbf{p}),T(\mathbf{x}',\mathbf{p}')] = 2(2\pi)^d\sin\left(\frac{\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{p}'}-\nabla_{\mathbf{x}'}\cdot\nabla_{\mathbf{p}}}{2}\right)\left[\delta_{\mathbf{x},\mathbf{x}'}\delta_{\mathbf{p},\mathbf{p}'}T(\mathbf{x},\mathbf{p})\right], \tag{35}
$$

which provides an identification of the structure constants of the Moyal algebra. From the basis of generators, any general element $O_F\in\mathfrak{g}$ can be written

$$
O_F = \int\frac{d^dxd^dp}{(2\pi)^d}F(\mathbf{x},\mathbf{p})T(\mathbf{x},\mathbf{p}), \tag{36}
$$

and so the Moyal Lie group, say G, then consists of elements

$$\exp(O_F) = \exp\left(\int \frac{d^dxd^dp}{(2\pi)^d} F(\mathbf{x},\mathbf{p})T(\mathbf{x},\mathbf{p})\right) \in G, \tag{37}$$

where $O_F \in \mathfrak{g}$ is an element of the Moyal algebra. The commutator (35) is then written

$$\begin{aligned}
[O_F, O_G] &= \int \frac{d^dxd^dp}{(2\pi)^d}\frac{d^dx'd^dp'}{(2\pi)^d} F(\mathbf{x},\mathbf{p})G(\mathbf{x}',\mathbf{p}')[T(\mathbf{x},\mathbf{p}),T(\mathbf{x}',\mathbf{p}')] \\
&= \int \frac{d^dxd^dp}{(2\pi)^d}\frac{d^dx'd^dp'}{(2\pi)^d} F(\mathbf{x},\mathbf{p})G(\mathbf{x}',\mathbf{p}') \times 2(2\pi)^d \sin\left(\frac{\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{p}'} - \nabla_{\mathbf{x}'}\cdot\nabla_{\mathbf{p}}}{2}\right)[\delta_{\mathbf{x},\mathbf{x}'}\delta_{\mathbf{p},\mathbf{p}'}T(\mathbf{x},\mathbf{p})] \\
&= \int \frac{d^dxd^dp}{(2\pi)^d} d^dx'd^dp' \left[2F(\mathbf{x},\mathbf{p})\sin\left(\frac{\overleftarrow{\nabla}_{\mathbf{x}}\cdot\overrightarrow{\nabla}_{\mathbf{p}'} - \overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{x}'}}{2}\right)G(\mathbf{x}',\mathbf{p}')\right]\delta_{\mathbf{x},\mathbf{x}'}\delta_{\mathbf{p},\mathbf{p}'}T(\mathbf{x},\mathbf{p}) \\
&= \int \frac{d^dxd^dp}{(2\pi)^d}\left[2F(\mathbf{x},\mathbf{p})\sin\left(\frac{\overleftarrow{\nabla}_{\mathbf{x}}\cdot\overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{x}}}{2}\right)G(\mathbf{x},\mathbf{p})\right]T(\mathbf{x},\mathbf{p}) \\
&= \int \frac{d^dxd^dp}{(2\pi)^d}\{F,G\}_M(\mathbf{x},\mathbf{p})T(\mathbf{x},\mathbf{p}) \\
&= O_{\{F,G\}_M},
\end{aligned} \tag{38}$$

where

$$\boxed{\{F,G\}_M = 2F\sin\left(\frac{\overleftarrow{\nabla}_{\mathbf{x}}\cdot\overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{x}}}{2}\right)G} \tag{39}$$

is known as the *Moyal bracket*, oftenly denoted by $\{\{F,G\}\}$ instead.

Aside on the Moyal Bracket: Restoring factors of $\hbar$, the Moyal bracket limits to the Poisson bracket in the semi-classical limit $\hbar \to 0$,

$$\begin{aligned}
\lim_{\hbar\to 0}\{F,G\}_M = \lim_{\hbar\to 0}\frac{2}{\hbar}F\sin\left(\frac{\hbar}{2}(\overleftarrow{\nabla}_{\mathbf{x}}\cdot\overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{x}})\right)G &= F(\overleftarrow{\nabla}_{\mathbf{x}}\cdot\overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{x}})G \\
&= \{F,G\}.
\end{aligned} \tag{40}$$

When $\hbar$ is absent, the semi-classical limit is written $\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{p}} \ll 1$. The Poisson algebra is not a subset of the Moyal algebra, it is rather a truncation of the bracket. Defining the *Moyal product*, $\star$, through

$$F \star G = F\exp\left(\frac{i}{2}(\overleftarrow{\nabla}_{\mathbf{x}}\cdot\overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{x}})\right)G, \tag{41}$$

permits a re-expression

$$\{F,G\}_M = \frac{1}{i}(F\star G - G\star F) = \frac{1}{i}[F,G]_\star \tag{42}$$

of the Moyal bracket, making it reminiscent of a commutator. This structure is a consequence of the Moyal product being the equivalent of operator multiplication in phase space. In essence, the Moyal algebra expresses the fact that phase space is non-commutative: $[\hat{x},\hat{p}] \neq 0$.

For what follows, if elements of the Moyal algebra are $F(\mathbf{x},\mathbf{p}) \in \mathfrak{g}$, then I define group action through the operator $U(\mathbf{x},\mathbf{p}) = \exp(-iF(\mathbf{x},\mathbf{p}))$. This ensures that

$$\begin{aligned} \mathrm{Ad}^*_{U=\exp(-iF)} f = U \star f \star U^{-1} &= f - i[F,f]_\star + \cdots \\ &= f + \{F,f\}_M + \cdots , \end{aligned} \tag{43}$$

is an expansion in Moyal brackets, without additional factors of i. On the dual Moyal algebra, then, I can define

$$\mathrm{ad}^*_F f = \{F,f\}_M, \tag{44}$$

such that $\mathrm{Ad}^*_{\exp(-iF)} f = \exp(\mathrm{ad}^*_F) f = U \star f \star U^{-1}$, and $\mathrm{ad}_F G = \{F,G\}_M$ is the Lie bracket.

2.2 First Quantized Approach

The Moyal algebra can also be derived from the first-quantized picture, where instead of representing operators $\hat{A}$ by fermion bilinears, we denote them as $\widehat{a}$ such that $\hat{A} = c_i^\dagger \langle i|\widehat{a}|j\rangle c_j$. This is the approach followed by Yuxuan's work [14], which I intend to focus on for a majority of these notes. As the commutator of two bilinears is also a bilinear, this is just as natural of a representation, and connects nicely with the derivation of the bosonized action from the coherent state path integral. The Wigner function for some first-quantized operator $\widehat{a}$ is given by the Wigner transformation

$$a(\mathbf{x},\mathbf{p}) = \int d^d y \left\langle \mathbf{x} - \frac{\mathbf{y}}{2} \middle| \widehat{a} \middle| \mathbf{x} + \frac{\mathbf{y}}{2} \right\rangle e^{i\mathbf{p}\cdot\mathbf{y}}, \qquad \text{(Wigner)} \tag{45}$$

which is analogous to that of (25). The inverse operation, known as the Weyl transformation, is straightforward to derive,

$$\begin{aligned} a\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\mathbf{p}\right) &= \int d^d y \left\langle \frac{\mathbf{x}+\mathbf{x}'}{2} - \frac{\mathbf{y}}{2} \middle| \widehat{a} \middle| \frac{\mathbf{x}+\mathbf{x}'}{2} + \frac{\mathbf{y}}{2} \right\rangle e^{i\mathbf{p}\cdot\mathbf{y}} \\ \iff \int \frac{d^d p}{(2\pi)^d}\, a\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\mathbf{p}\right) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} &= \int \frac{d^d y d^d p}{(2\pi)^d} \left\langle \frac{\mathbf{x}+\mathbf{x}'}{2} - \frac{\mathbf{y}}{2} \middle| \widehat{a} \middle| \frac{\mathbf{x}+\mathbf{x}'}{2} + \frac{\mathbf{y}}{2} \right\rangle e^{i\mathbf{p}\cdot(\mathbf{y}+\mathbf{x}-\mathbf{x}')} \\ &= \int d^d y \left\langle \frac{\mathbf{x}+\mathbf{x}'}{2} - \frac{\mathbf{y}}{2} \middle| \widehat{a} \middle| \frac{\mathbf{x}+\mathbf{x}'}{2} + \frac{\mathbf{y}}{2} \right\rangle \delta(\mathbf{y}+\mathbf{x}-\mathbf{x}'), \\ \iff \langle \mathbf{x}|\,\widehat{a}\,|\mathbf{x}'\rangle &= \int \frac{d^d p}{(2\pi)^d}\, a\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\mathbf{p}\right) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}. \qquad \text{(Weyl)} \end{aligned} \tag{46}$$

A simple relation is the trace of the product of operators

$$\mathrm{tr}[\widehat{a}\widehat{b}] = \int \frac{d^d x d^d p}{(2\pi)^d} a(\mathbf{x},\mathbf{p}) b(\mathbf{x},\mathbf{p}), \tag{47}$$

which is nothing but the inner product between their Wigner functions. This is also straightforward to show,

$$\begin{aligned} \int \frac{d^d x d^d p}{(2\pi)^d} a(\mathbf{x},\mathbf{p}) b(\mathbf{x},\mathbf{p}) &= \int \frac{d^d x d^d p}{(2\pi)^d} d^d y d^d y' e^{i\mathbf{p}\cdot(\mathbf{y}+\mathbf{y}')} \left\langle \mathbf{x} - \frac{\mathbf{y}}{2} \middle| \widehat{a} \middle| \mathbf{x} + \frac{\mathbf{y}}{2} \right\rangle \left\langle \mathbf{x} - \frac{\mathbf{y}'}{2} \middle| \widehat{b} \middle| \mathbf{x} + \frac{\mathbf{y}'}{2} \right\rangle \\ &= \int d^d x d^d y \left\langle \mathbf{x} - \frac{\mathbf{y}}{2} \middle| \widehat{a} \middle| \mathbf{x} + \frac{\mathbf{y}}{2} \right\rangle \left\langle \mathbf{x} + \frac{\mathbf{y}}{2} \middle| \widehat{b} \middle| \mathbf{x} - \frac{\mathbf{y}}{2} \right\rangle \\ &= \int d^d z d^d z' \; \langle \mathbf{z}|\,\widehat{a}\,|\mathbf{z}'\rangle \langle \mathbf{z}'|\,\widehat{b}\,|\mathbf{z}\rangle = \mathrm{tr}[\widehat{a}\widehat{b}] \quad \square \end{aligned} \tag{48}$$

What is perhaps less straightforward to show, is that

$$\text{if} \quad \widehat{c} = [\widehat{a},\widehat{b}] \quad \text{then} \quad c(\mathbf{x},\mathbf{p}) = a(\mathbf{x},\mathbf{p}) \star b(\mathbf{x},\mathbf{p}) - b(\mathbf{x},\mathbf{p}) \star a(\mathbf{x},\mathbf{p}) = i\{a(\mathbf{x},\mathbf{p}), b(\mathbf{x},\mathbf{p})\}_M, \tag{49}$$

i.e. the Wigner function of the commutator of two operators is (i times) the Moyal bracket of the their associated Wigner functions. This is shown in detail in below.

Proof of (49): From the Wigner representation (45), we have that

$$\begin{aligned} c(\mathbf{x},\mathbf{p}) &= \int d^dy\ \left\langle \mathbf{x}-\frac{\mathbf{y}}{2}\middle|\widehat{a}\widehat{b}\middle|\mathbf{x}+\frac{\mathbf{y}}{2}\right\rangle e^{i\mathbf{p}\cdot\mathbf{y}} - \{a\leftrightarrow b\} \\ &= \int d^dzd^dy\ \left\langle \mathbf{x}-\frac{\mathbf{y}}{2}|\widehat{a}|\mathbf{z}\right\rangle \left\langle \mathbf{z}|\widehat{b}|\mathbf{x}+\frac{\mathbf{y}}{2}\right\rangle e^{i\mathbf{p}\cdot\mathbf{y}} - \{a\leftrightarrow b\}, \end{aligned} \tag{50}$$

after inserting a completeness relation. I will now replace the two brackets with their Weyl representation (46),

$$\begin{aligned} c(\mathbf{x},\mathbf{p}) &= \int d^dzd^dy \frac{d^dp'd^dp''}{(2\pi)^{2d}}\ e^{i\mathbf{p}\cdot\mathbf{y}}e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y}/2-\mathbf{z})}e^{i\mathbf{p}''\cdot(\mathbf{z}-\mathbf{x}-\mathbf{y}/2)}a\left(\frac{\mathbf{x}}{2}+\frac{\mathbf{z}-\mathbf{y}/2}{2},\mathbf{p}'\right)b\left(\frac{\mathbf{x}}{2}+\frac{\mathbf{z}+\mathbf{y}/2}{2},\mathbf{p}''\right) \\ &\qquad - \{a\leftrightarrow b\} \\ &= \int d^dzd^dy \frac{d^dp'd^dp''}{(2\pi)^{2d}}\ e^{i\mathbf{p}\cdot\mathbf{y}}e^{i(\mathbf{p}+\mathbf{p}')\cdot(\mathbf{x}-\mathbf{y}/2-\mathbf{z})}e^{i(\mathbf{p}+\mathbf{p}'')\cdot(\mathbf{z}-\mathbf{x}-\mathbf{y}/2)}a\left(\frac{\mathbf{x}}{2}+\frac{\mathbf{z}-\mathbf{y}/2}{2},\mathbf{p}+\mathbf{p}'\right)b\left(\frac{\mathbf{x}}{2}+\frac{\mathbf{z}+\mathbf{y}/2}{2},\mathbf{p}+\mathbf{p}''\right) \\ &\qquad - \{a\leftrightarrow b\} \\ &= \int d^dud^du' \frac{d^dp'd^dp''}{(2\pi)^{2d}}\ e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{u}')}e^{i\mathbf{p}''\cdot(\mathbf{u}-\mathbf{x})}a\left(\frac{\mathbf{x}}{2}+\frac{\mathbf{u}}{2},\mathbf{p}+\mathbf{p}'\right)b\left(\frac{\mathbf{x}}{2}+\frac{\mathbf{u}'}{2},\mathbf{p}+\mathbf{p}''\right) - \{a\leftrightarrow b\}, \end{aligned} \tag{51}$$

where the second step followed from the shift $\mathbf{p}'\to\mathbf{p}'+\mathbf{p}$ (likewise for $\mathbf{p}''$) and the last step followed from the substitution $\mathbf{u}=\mathbf{z}-\mathbf{y}/2,\ \mathbf{u}'=\mathbf{z}+\mathbf{y}/2$ (unit Jacobian). Using the fact that

$$f(\mathbf{x}+\mathbf{x}_0,\mathbf{y}+\mathbf{y}_0) = \exp(\mathbf{x}_0\cdot\nabla_{\mathbf{x}})\exp(\mathbf{y}_0\cdot\nabla_{\mathbf{y}})f(\mathbf{x},\mathbf{y}), \tag{52}$$

I find

$$\begin{aligned} c(\mathbf{x},\mathbf{p}) = a\left(\frac{\mathbf{x}}{2},\mathbf{p}\right)\Big[\int d^dud^du' \frac{d^dp'd^dp''}{(2\pi)^{2d}}\ \exp\left(i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{u}')\right)\exp\left(i\mathbf{p}''\cdot(\mathbf{u}-\mathbf{x})\right) \\ \times\exp\left((\mathbf{u}/2)\cdot\nabla^a_{\mathbf{x}}\right)\exp\left(\mathbf{p}'\cdot\nabla^a_{\mathbf{p}}\right)\exp\left((\mathbf{u}'/2)\cdot\nabla^b_{\mathbf{x}}\right)\exp\left(\mathbf{p}''\cdot\nabla^b_{\mathbf{p}}\right)\Big]b\left(\frac{\mathbf{x}}{2},\mathbf{p}\right) - \{a\leftrightarrow b\}, \end{aligned} \tag{53}$$

where the superscript of the gradient operator is the function it acts on. I can now collect exponentials and perform the integrations,

$$\begin{aligned} c(\mathbf{x},\mathbf{p}) &= a\left(\frac{\mathbf{x}}{2},\mathbf{p}\right)\Big[\int d^du\ \exp\left(\frac{\mathbf{u}}{2}\cdot\nabla^a_{\mathbf{x}}\right)\int d^du'\ \exp\left(\frac{\mathbf{u}'}{2}\cdot\nabla^b_{\mathbf{x}}\right) \\ &\qquad\times\int\frac{d^dp'}{(2\pi)^d}\ \exp\left(i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{u}'-i\nabla^a_{\mathbf{p}})\right)\int\frac{d^dp''}{(2\pi)^d}\ \exp\left(i\mathbf{p}''\cdot(\mathbf{u}-\mathbf{x}-i\nabla^b_{\mathbf{p}})\right)\Big]b\left(\frac{\mathbf{x}}{2},\mathbf{p}\right) - \{a\leftrightarrow b\} \\ &= a\left(\frac{\mathbf{x}}{2},\mathbf{p}\right)\Big[\int d^du\ \exp\left(\frac{\mathbf{u}}{2}\cdot\nabla^a_{\mathbf{x}}\right)\int d^du'\ \exp\left(\frac{\mathbf{u}'}{2}\cdot\nabla^b_{\mathbf{x}}\right)\delta\left(\mathbf{x}-\mathbf{u}'-i\nabla^a_{\mathbf{p}}\right)\delta\left(\mathbf{u}-\mathbf{x}-i\nabla^b_{\mathbf{p}}\right)\Big]b\left(\frac{\mathbf{x}}{2},\mathbf{p}\right) \\ &\qquad - \{a\leftrightarrow b\} \\ &= a\left(\frac{\mathbf{x}}{2},\mathbf{p}\right)\left[\exp\left(\frac{(\mathbf{x}+i\nabla^b_{\mathbf{p}})}{2}\cdot\nabla^a_{\mathbf{x}}\right)\exp\left(\frac{(\mathbf{x}-i\nabla^a_{\mathbf{p}})}{2}\cdot\nabla^b_{\mathbf{x}}\right)\right]b\left(\frac{\mathbf{x}}{2},\mathbf{p}\right) - \{a\leftrightarrow b\} \\ &= a\left(\frac{\mathbf{x}}{2},\mathbf{p}\right)\exp\left(\frac{\mathbf{x}}{2}\cdot\nabla^a_{\mathbf{x}}\right)\left[\exp\left(\frac{i}{2}\left(\nabla^a_{\mathbf{x}}\cdot\nabla^b_{\mathbf{p}}-\nabla^b_{\mathbf{x}}\cdot\nabla^a_{\mathbf{p}}\right)\right)\right]\exp\left(\frac{\mathbf{x}}{2}\cdot\nabla^b_{\mathbf{x}}\right)b\left(\frac{\mathbf{x}}{2},\mathbf{p}\right) - \{a\leftrightarrow b\}, \end{aligned} \tag{54}$$

where from (52), I can shift the coordinates of a,b by $\mathbf{x}/2$, resulting in

$$\begin{aligned} c(\mathbf{x},\mathbf{p}) &= a\left(\mathbf{x},\mathbf{p}\right)\exp\left(\frac{i}{2}\left(\overleftarrow{\nabla}_{\mathbf{x}}\cdot\overrightarrow{\nabla}_{\mathbf{p}}-\overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{x}}\right)\right)b\left(\mathbf{x},\mathbf{p}\right) - \{a\leftrightarrow b\} \\ &= a(\mathbf{x},\mathbf{p})\star b(\mathbf{x},\mathbf{p}) - b(\mathbf{x},\mathbf{p})\star a(\mathbf{x},\mathbf{p}), \quad \square \end{aligned} \tag{55}$$

as desired!

3 Bosonized Action From Coherent State Path-Integral

This section is covered in the appendix of [14], and in these notes I will expand on the details. Denote the ground state $|\mathrm{FS}\rangle$ as a spherical Fermi sea filled up to the chemical potential, i.e. $|\mathrm{FS}\rangle = \prod_{|\mathbf{k}|<k_F} c^\dagger_{\mathbf{k}} |0\rangle$. Excited states consist of particle-hole excitations, which are generated by fermion bilinears $\sim ic^\dagger_i c_j$ acting on $|\mathrm{FS}\rangle$,

$$|\mathcal{U}\rangle = \exp(i\phi_{ij}c^\dagger_i c_k)\,|\mathrm{FS}\rangle \equiv \hat{\mathcal{U}}\,|\mathrm{FS}\rangle\,. \tag{56}$$

Here, $|\mathcal{U}\rangle$ is a coherent state in the many-body Hilbert space,[3] and ϕ is an arbitrary function to be eventually identified as the bosonic degree of freedom. The space of all such coherent states $|\mathcal{U}\rangle$ is known as the *orbit* of $|\mathrm{FS}\rangle$ under action of the Moyal group.

The coherent state describes, through the function ϕ, the *collective* low energy rearrangement of particle occupation near the Fermi level (like a deformation of the FS shape). From a microscopic point of view, this fluctuation of occupation is mediated by particle-hole (quasi-particle) excitations. As a way of characterizing the space of states, I will develop a coherent state path integral for the bosonic field ϕ governed by some prescribed Hamiltonian $\hat{H}$. This naturally provides an alternative (field-theoretic) formulation of Landau's Fermi liquid theory in its description of the FS and its quasi-particle structure, and the two should agree in certain limits.

3.1 Derivation

Good references are the textbooks [1, 9], which derive the coherent state path integral for spin degrees of freedom. The Haar measure, $d\mathcal{U}$, assigns an invariant volume to locally compact Lie groups, and consequently defines an integral for functions on them. Since the Haar measure is invariant under left and right multiplication of fixed group elements[4], then for $\mathcal{U},\mathcal{U}' \in G$,

$$\hat{\mathcal{U}}'\left(\int d\mathcal{U}\,|\mathcal{U}\rangle\,\langle\mathcal{U}|\right) = \int d\mathcal{U}\,|\mathcal{U}'\mathcal{U}\rangle\,\langle\mathcal{U}| = \int d\mathcal{U}\,|\mathcal{U}'(\mathcal{U}')^{-1}\mathcal{U}\rangle\,\langle(\mathcal{U}')^{-1}\mathcal{U}| = \left(\int d\mathcal{U}\,|\mathcal{U}\rangle\,\langle\mathcal{U}|\right)\hat{\mathcal{U}}', \tag{57}$$

and the quantity $\int d\mathcal{U}\,|\mathcal{U}\rangle\,\langle\mathcal{U}|$ commutes with all other group elements. From Schur's Lemma[5], I deduce the identity

$$1 = \int d\mathcal{U}\,|\mathcal{U}\rangle\,\langle\mathcal{U}|\,, \tag{58}$$

which is useful for construction of the path integral

$$\mathcal{Z} = \left\langle \mathcal{U}_F|e^{-i\hat{H}\Delta t}|\mathcal{U}_i\right\rangle \approx \lim_{N\to\infty}\left\langle \mathcal{U}_F|\left(I - i\varepsilon\hat{H}\right)^N|\mathcal{U}_i\right\rangle, \tag{59}$$

where $\varepsilon \equiv \Delta t/N$ is used to break the path into N steps. Inserting the identity (58) between each copy of $(I-\varepsilon\hat{H})$, I will be left with terms of the form

$$\begin{aligned}\langle\mathcal{U}_{j+1}|\,(I-i\varepsilon\hat{H})\,|\mathcal{U}_j\rangle &= \langle\mathcal{U}_{j+1}|\mathcal{U}_j\rangle - \langle\mathcal{U}_{j+1}|\,(i\varepsilon\hat{H})\,|\mathcal{U}_j\rangle \\ &= 1 + \langle\mathcal{U}_{j+1}|\mathcal{U}_j\rangle - \langle\mathcal{U}_j|\mathcal{U}_j\rangle - \langle\mathcal{U}_{j+1}|\,(i\varepsilon\hat{H})\,|\mathcal{U}_j\rangle \\ &\approx \exp\left(\varepsilon\left(\frac{\langle\mathcal{U}_{j+1}|\mathcal{U}_j\rangle - \langle\mathcal{U}_j|\mathcal{U}_j\rangle}{\varepsilon} - \langle\mathcal{U}_{j+1}|\,(i\hat{H})\,|\mathcal{U}_j\rangle\right)\right),\end{aligned} \tag{60}$$

where $j \to j+1$ corresponds to a time shift $t \to t+\varepsilon$. Hence,

$$\begin{aligned}\mathcal{Z} &= \lim_{N\to\infty}\int\prod_{j=0}^{N-1} d\mathcal{U}_j\exp\left(\varepsilon\left(\frac{\langle\mathcal{U}_{j+1}|\mathcal{U}_j\rangle - \langle\mathcal{U}_j|\mathcal{U}_j\rangle}{\varepsilon} - \langle\mathcal{U}_{j+1}|\,(i\hat{H})\,|\mathcal{U}_j\rangle\right)\right) \\ &\approx \int\mathcal{D}\mathcal{U}(t)\exp\left(\int_t\left(\langle\partial_t\mathcal{U}|\mathcal{U}\rangle - \langle\mathcal{U}|\,(i\hat{H})\,|\mathcal{U}\rangle\right)\right) \\ &\approx \int\mathcal{D}\mathcal{U}(t)\exp\left(\int_t\langle\mathcal{U}|\left(-\partial_t - i\hat{H}\right)|\mathcal{U}\rangle\right)\end{aligned} \tag{61}$$

[3] Note that $\{|\mathcal{U}\rangle\}$ is an overcomplete basis, with $\langle\mathcal{U}|\mathcal{U}\rangle = 1,\ \langle\mathcal{U}|\mathcal{U}'\rangle \neq 1$.

[4] It is left-invariant by definition, and also right-invariant if the group is compact.

[5] An operator $\hat{A}$ is proportional to the unit identity iff $\hat{A}$ commutes with all the representation matrices of an irreducible group representation.

where $\mathcal{DU} \equiv \lim_{N\to\infty} \prod_{j=0}^{N-1} d\mathcal{U}_j$, and the last approximation followed from an IBP. In terms of the $|\text{FS}\rangle$ ground state, this can be rewritten in the form

$$\mathcal{Z} = \int \mathcal{DU}(t) \exp\left(\int_t \langle \text{FS}| \hat{\mathcal{U}}^{-1} \left(-\partial_t - i\hat{H} \right) \hat{\mathcal{U}} \, |\text{FS}\rangle \right), \tag{62}$$

as quoted in [14].

Sandwhiched between the ground states is

$$\hat{\mathcal{U}}^{-1} \left(-\partial_t - i\hat{H} \right) \hat{\mathcal{U}} = \hat{\mathcal{U}}^{-1}(-\partial_t)\hat{\mathcal{U}} + \hat{\mathcal{U}}^{-1}(-i\hat{H})\hat{\mathcal{U}}. \tag{63}$$

The general elements (operators) in the Lie algebra of fermion bilinears are by definition represented as linear combinations of the generators $c_i^\dagger c_j$, such that $\hat{A} = c_i^\dagger a_{ij} c_j$. This means (63) is an expansion of commutators of fermion bilinears, and the trick is that for any two bilinears $\hat{A}$ and $\hat{B}$,

$$\begin{aligned} [\hat{A}, \hat{B}] = [c_i^\dagger a_{ik} c_k, c_l^\dagger b_{lj} c_j] = a_{ik} b_{lj} [c_i^\dagger c_k, c_l^\dagger c_j] = a_{ik} b_{lj} (\delta_{kl} c_i^\dagger c_j - \delta_{ij} c_l^\dagger c_k) &= c_i^\dagger a_{ik} b_{kj} c_j - c_l^\dagger b_{lj} a_{jk} c_k \\ &= c_i^\dagger (a_{ik} b_{kj} - b_{ik} a_{kj}) c_j \\ &= c_i^\dagger [\widehat{a}, \widehat{b}]_{ij} c_j, \end{aligned} \tag{64}$$

where $\widehat{a}$ are first quantized operators such that $\hat{A} = c_i^\dagger \langle i|\widehat{a}|j\rangle c_j$. This means that

$$\begin{aligned} \hat{\mathcal{U}}^{-1}(-i\hat{H})\hat{\mathcal{U}} = -ic_i^\dagger \varepsilon_{ij} c_j - [i\phi_{ij} c_i^\dagger c_j, -ic_k^\dagger \varepsilon_{kl} c_l] + \mathcal{O}((\phi_{ij})^2) &= c^\dagger \left[-i\widehat{\varepsilon} + [i\widehat{\phi}, i\widehat{\varepsilon}] + \cdots \right]_{ij} c_j \\ &= c_i^\dagger c_j \left[\widehat{U}^{-1}(-i\widehat{\varepsilon})\widehat{U} \right]_{ij}, \end{aligned} \tag{65}$$

where I defined $\widehat{U} \equiv \exp(i\widehat{\phi})$, and noted how in this representation the fermion operators can be freely moved around. For term with the time-derivative, I must be careful; note that

$$\partial_t e^{\hat{O}} = \int_0^1 du \; e^{\hat{O}(1-u)} (\partial_t \hat{O}) e^{\hat{O}u}, \tag{66}$$

for some operator $\hat{O}$ [9]. This means that

$$\begin{aligned} \hat{\mathcal{U}}^{-1}\partial_t\hat{\mathcal{U}} = \exp(-i\phi_{ij}c_i^\dagger c_j)\partial_t \exp(-i\phi_{ij}c_i^\dagger c_j) &= \exp(-i\phi_{ij}c_i^\dagger c_j) \int_0^1 du \; \exp(i(1-u)\phi_{ij}c_i^\dagger c_j)\, c_i^\dagger (i\partial_t\phi_{ij}) c_j \; \exp(iu\phi_{ij}c_i^\dagger c_j) \\ &= \int_0^1 du \; \exp(-iu\phi_{ij}c_i^\dagger c_j)\, c_i^\dagger (i\partial_t\phi_{ij}) c_j \; \exp(iu\phi_{ij}c_i^\dagger c_j) \\ &= \int_0^1 du \; \left[c_i^\dagger (i\partial_t\phi_{ij}) c_j - u[i\phi_{kl}c_k^\dagger c_l, i\partial_t\phi_{ij}c_i^\dagger c_j] + \cdots \right] \\ &= c_i^\dagger (i\partial_t\phi_{ij}) c_j - \frac{1}{2} \left[c_k^\dagger [i\widehat{\phi}]_{kl} c_l,\; c_i^\dagger [i\partial_t\widehat{\phi}]_{ij} c_j \right] + \cdots \\ &= c_i^\dagger \left[\partial_t(i\widehat{\phi}) - \frac{1}{2} \left[i\widehat{\phi},\; \partial_t(i\widehat{\phi}) \right] + \cdots \right]_{ij} c_j, \end{aligned} \tag{67}$$

where the last step follows from (64).[6] At this point I can back track through the same steps inside the brackets, just without any fermionic operators this time. I can immediately deduce

$$\hat{\mathcal{U}}^{-1}\partial_t\hat{\mathcal{U}} = c_i^\dagger \left[e^{-i\widehat{\phi}} \partial_t e^{i\widehat{\phi}} \right]_{ij} c_j = c_i^\dagger \left[\widehat{U}^{-1} \partial_t \widehat{U} \right]_{ij} c_j, \tag{68}$$

which is a lovely result. Inputting these results into the partition function (62) above, I find the action

$$S = \int_t \langle \text{FS}|\, c_i^\dagger c_j \,|\text{FS}\rangle \left[\widehat{U}^{-1}(i\partial_t - \widehat{\varepsilon})\widehat{U} \right]_{ij}. \tag{69}$$

[6] I also could've quoted the Wikipedia result $e^{-X}de^X = dX - \frac{1}{2!}[X, dX] + \cdots$ where d is the differential operator [12].

Defining the one-particle density matrix for the Fermi surface ground state $[\widehat{f}_0]_{ji} = \left\langle \mathrm{FS}|c_i^\dagger c_j|\mathrm{FS}\right\rangle$, the action can be written as a trace over first quantized operators,

$$S = \int_t \mathrm{tr}\left[\widehat{f}_0 \widehat{U}^{-1}(i\partial_t - \widehat{\varepsilon})\widehat{U}\right] = \int_t \mathrm{tr}\left[\widehat{f}_0 \widehat{U}^{-1}(i\partial_t)\widehat{U} - \widehat{f}\widehat{\varepsilon}\right], \tag{70}$$

where $[\widehat{f}]_{ji} = [\widehat{U}\widehat{f}_0\widehat{U}^{-1}]_{ij}$ is the one-particle density matrix for the coherent state.

The first trace can be evaluated from the result $e^{-X}de^{X} = dX - \frac{1}{2!}[X, dX] + \cdots$, in which case

$$\mathrm{tr}\left[\widehat{f}_0 \widehat{U}^{-1}(i\partial_t)\widehat{U}\right] = \mathrm{tr}\left[\widehat{f}_0\left((i\partial_t)(i\widehat{\phi}) - \frac{1}{2}[i\widehat{\phi}, (i\partial_t)(i\widehat{\phi})] + \cdots\right)\right], \tag{71}$$

which can be simplified by noting that trace of a product of first-quantized operators is the inner product of their Wigner functions (47), and the Wigner function of a commutator of first-quantized operators is (i times) the Moyal bracket of the associated Wigner functions (49):

$$\begin{aligned}\mathrm{tr}\left[\widehat{f}_0 \widehat{U}^{-1}(i\partial_t)\widehat{U}\right] &= \int \frac{d^dxd^dp}{(2\pi)^d}\left(f_0(\mathbf{p})\left((i\partial_t)(i\phi(\mathbf{x},\mathbf{p},t)) - \frac{i}{2}\{i\phi(\mathbf{x},\mathbf{p},t), (i\partial_t)(i\phi(\mathbf{x},\mathbf{p},t))\}_M + \cdots\right)\right)\\ &= \int \frac{d^dxd^dp}{(2\pi)^d}\left(f_0(\mathbf{p})U(\mathbf{x},\mathbf{p},t)^{-1} \star i\partial_t U(\mathbf{x},\mathbf{p},t)\right),\end{aligned} \tag{72}$$

where $U(\mathbf{x},\mathbf{p},t) = \exp(i\phi(\mathbf{x},\mathbf{p},t))$. Similarly,

$$\mathrm{tr}[\widehat{f}\widehat{\varepsilon}] = \int \frac{d^dxd^dp}{(2\pi)^d} f(\mathbf{x},\mathbf{p})\varepsilon(\mathbf{p}), \tag{73}$$

where

$$\begin{aligned} f(\mathbf{x},\mathbf{p}) = \mathrm{Ad}_U^* f_0(\mathbf{p}) &= U(\mathbf{x},\mathbf{p}) \star f_0(\mathbf{p}) \star U(\mathbf{x},\mathbf{p})^{-1}\\ &= f_0(\mathbf{p}) - \{\phi(\mathbf{x},\mathbf{p}), f_0(\mathbf{p})\}_M + \frac{1}{2!}\{\phi(\mathbf{x},\mathbf{p}), \{\phi(\mathbf{x},\mathbf{p}), f_0(\mathbf{p})\}_M\}_M + \cdots,\end{aligned} \tag{74}$$

and $\varepsilon(\mathbf{p})$ is the Wigner function of $\widehat{\epsilon}$.

I conclude with the action

$$\boxed{S = \int dt \frac{d^dxd^dp}{(2\pi)^d}\left(f_0(\mathbf{p})U(\mathbf{x},\mathbf{p},t)^{-1} \star i\partial_t U(\mathbf{x},\mathbf{p},t) - f(\mathbf{x},\mathbf{p})\varepsilon(\mathbf{p})\right)} \tag{75}$$

exactly as that quoted in the appendix of [14]. In terms of inner products, the action can be written

$$S = S_{\mathrm{WZW}} + S_H, \tag{76}$$

where

$$S_{\mathrm{WZW}} = \int dt \left\langle f_0(\mathbf{p}), U(\mathbf{x},\mathbf{p},t)^{-1} \star i\partial_t U(\mathbf{x},\mathbf{p},t)\right\rangle, \qquad S_H = -\int dt \left\langle f(\mathbf{x},\mathbf{p}), \varepsilon(\mathbf{p})\right\rangle. \tag{77}$$

This is done for non-interacting fermions. In the presence of interactions, S_H contains an additional term composed of Landau parameters (LPs) for the Fermi liquid [14]. Yuxuan does not include LPs in his work [14], in which he develops the bosonized theory in weak field, but he does include them in [11] in an in-depth analysis of the dHvA effect.

3.2 Quantum Boltzmann Equation

The equation of motion for the action (76) should be the (quantum) Boltzmann equation, limiting to the semi-classical Boltzmann equation (22) as $\hbar \to 0$. Noting that

$$\langle f, \varepsilon\rangle = \langle \mathrm{Ad}_U^* f_0, \varepsilon\rangle = \langle f_0, \mathrm{Ad}_{U^{-1}} \varepsilon\rangle = \left\langle f_0, U^{-1} \star \varepsilon \star U\right\rangle, \tag{78}$$

then the action can be re-expressed as

$$S = \int dt \left\langle f_0, U^{-1} \star (i\partial_t - \varepsilon\star) U \right\rangle \iff 0 = \delta S = \int dt \left\langle f_0, \delta \left[U^{-1} \star i\partial_t U\right] - \delta \left[U^{-1} \star \varepsilon \star U\right] \right\rangle, \tag{79}$$

and the implication is the assumption of stationary action. I will determine the variations by shifting the dynamic operators under Moyal group action $U \to U' = e^{\alpha} \star U$, for $\alpha = \alpha(\mathbf{x}, \mathbf{p}, t) \in \mathfrak{g}$ assumed to small. This follows the steps taken in [3], albeit with Moyal algebra instead of Poisson algebra.

Computing Variations: The first term of the action shifts under $U \to U'$ as

$$\begin{aligned} U^{-1} \star i\partial_t U \quad \to \quad (e^{\alpha} \star U)^{-1} \star i\partial_t (e^{\alpha} \star U) = U^{-1} \star e^{-\alpha} \star (i\partial_t e^{\alpha} \star U + e^{\alpha} \star i\partial_t U) &= U^{-1} \star i\partial_t U + U^{-1} \star e^{-\alpha} \star i\partial_t e^{\alpha} \star U \\ &\equiv U^{-1} \star i\partial_t U + \delta \left[U^{-1} \star i\partial_t U\right], \end{aligned} \tag{80}$$

where I made use of the fact that $(U \star V)^{-1} = V^{-1} \star U^{-1}$, and identified the variation

$$\begin{aligned} \delta \left[U^{-1} \star i\partial_t U\right] = U^{-1} \star \left(e^{-\alpha} \star i\partial_t e^{\alpha}\right) \star U &= U^{-1} \star \left(i\partial_t \alpha - \frac{1}{2}[\alpha, i\partial_t \alpha]_\star + \mathcal{O}(\alpha^3) \right) \star U \\ &= U^{-1} \star i\partial_t \alpha \star U + \mathcal{O}(\alpha^2), \end{aligned} \tag{81}$$

to linear order α. The same can be done for the second term of the action,

$$\begin{aligned} U^{-1} \star \varepsilon \star U \quad \to \quad (e^{\alpha} \star U)^{-1} \star \varepsilon \star (e^{\alpha} \star U) = U^{-1} \star (e^{-\alpha} \star \varepsilon \star e^{\alpha}) \star U &= U^{-1} \star (\varepsilon - [\alpha, \varepsilon]_\star + \mathcal{O}(\alpha^2)) \star U \\ &\equiv U^{-1} \star \varepsilon \star U + \delta \left[U^{-1} \star \varepsilon \star U\right], \end{aligned} \tag{82}$$

where

$$\begin{aligned} \delta \left[U^{-1} \star \varepsilon \star U\right] &= -U^{-1} \star [\alpha, \varepsilon]_\star \star U + \mathcal{O}(\alpha^2) \\ &= U^{-1} \star (-i\{\alpha, \varepsilon\}_M) \star U + \mathcal{O}(\alpha^2) \end{aligned} \tag{83}$$

is the variation to linear order in α.

The stationary action condition $0 = \delta S$ becomes

$$\begin{aligned} 0 = \int dt \left\langle f_0, U^{-1} \star (i\partial_t \alpha + i\{\alpha, \varepsilon\}_M) \star U \right\rangle = \int dt \left\langle f, \partial_t \alpha + \{\alpha, \varepsilon\}_M \right\rangle &= \int dt \Big(\langle f, \partial_t \alpha \rangle - \langle f, \mathrm{ad}_\varepsilon\, \alpha \rangle \Big) \\ &= \int dt \Big(- \langle \partial_t f, \alpha \rangle + \langle \mathrm{ad}^*_\varepsilon\, f, \alpha \rangle \Big) \\ &= - \int dt \left\langle \partial_t f - \mathrm{ad}^*_\varepsilon\, f, \alpha \right\rangle, \end{aligned} \tag{84}$$

where I recalled my definition of the coadjoint (44), noted its defining property $\langle f, \mathrm{ad}_\varepsilon\, \alpha \rangle = - \langle \mathrm{ad}^*_\varepsilon\, f, \alpha \rangle$, and used IBP to write $\langle f, \partial_t \alpha \rangle = - \langle \partial_t f, \alpha \rangle$. This relation must hold $\forall \alpha$, and so

$$0 = \partial_t f - \mathrm{ad}^*_\varepsilon\, f \iff \boxed{\partial_t f = \mathrm{ad}^*_\varepsilon\, f = \{\varepsilon, f\}_M} \tag{85}$$

which is the quantum Boltzmann equation.[7] In the semiclassical limit, the Moyal bracket becomes the Poisson bracket, and I find the semi-classical Liouville equation (23).

[7] It's really the Liouville equation, since I omitted interactions.

3.3 Gaussian Action in the Semiclassical Limit / Momentum Space 2-Point Function

In this section, I will expand the bosonized action (76) in the bosonic parameter ϕ to Gaussian order in 2d, assuming free fermions $\varepsilon = \varepsilon(|\mathbf{p}|)$ and the semi-classical limit $\hbar \to 0$. Note that the one-particle density matrix f_0 is simply $f_0(\mathbf{p}) = \Theta(p_F - |\mathbf{p}|)$, indicating a filled spherical Fermi sea of radius p_F. For the Wess-Zumino-Witten term, I find

$$\begin{aligned} S_{\text{WZW}} = \int dt \left\langle f_0, U^{-1} \star i\partial_t U \right\rangle &\xrightarrow{\hbar\to 0} \int dt \frac{d^2x d^2p}{(2\pi)^2} f_0 \left(i\partial_t (i\phi) - \frac{i}{2}\left\{ i\phi, i\partial_t (i\phi)\right\}\right) + \mathcal{O}(\phi^3) \\ &= -\frac{1}{2}\int dt \frac{d^2x d^2p}{(2\pi)^2} f_0 \left\{ \phi, \dot{\phi} \right\} + \mathcal{O}(\phi^3) \end{aligned} \tag{86}$$

where I replaced Moyal brackets with Poisson brackets, and noted that the term linear in ϕ is a total time derivative (since f_0 is time-independent). Expanding the bracket and using IBP,

$$\begin{aligned} S_{\text{WZW}} &= -\frac{1}{2}\int dt \frac{d^2x d^2p}{(2\pi)^2} f_0 \left(\nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}}\dot{\phi} - \nabla_{\mathbf{p}}\phi \cdot \nabla_{\mathbf{x}}\dot{\phi} \right) + \mathcal{O}(\phi^3) \\ &= \frac{1}{2}\int dt \frac{d^2x d^2p}{(2\pi)^2} \dot{\phi} \left(\nabla_{\mathbf{p}} \cdot (f_0 \nabla_{\mathbf{x}}\phi) - \nabla_{\mathbf{x}} \cdot (f_0 \nabla_{\mathbf{p}}\phi) \right) + \mathcal{O}(\phi^3) \\ &= \frac{1}{2}\int dt \frac{d^2x d^2p}{(2\pi)^2} \dot{\phi} \nabla_{\mathbf{p}} f_0 \cdot \nabla_{\mathbf{x}}\phi + \mathcal{O}(\phi^3), \end{aligned} \tag{87}$$

where since

$$\nabla_{\mathbf{p}} f_0 = \nabla_{\mathbf{p}} \Theta(p_F - |\mathbf{p}|) = -\delta(|\mathbf{p}| - p_F)\hat{n}, \tag{88}$$

after defining $\hat{n} = \mathbf{p}/|\mathbf{p}|$ to be the unit vector normal to the FS, I find

$$S_{\text{WZW}} = -\frac{p_F}{2}\int dt \frac{d^2x d\theta}{(2\pi)^2} \dot{\phi} \left(\hat{n} \cdot \nabla_{\mathbf{x}}\phi \right) + \mathcal{O}(\phi^3), \tag{89}$$

for the WZW term. Note that I used the delta function to collapse the radial momentum integration, leaving just an integration over the angular variable $\theta = \tan^{-1}(p_y/p_x)$ and picking up a factor of p_F from the measure.

In a similar manner, the Hamiltonian term takes the form

$$S_H = -\int dt \left\langle f_0, U^{-1} \star \varepsilon \star U \right\rangle \xrightarrow{\hbar\to 0} -\int dt \frac{d^2x d^2p}{(2\pi)^2} f_0 \left(\varepsilon + \{\phi, \varepsilon\} + \frac{1}{2}\{\phi, \{\phi, \varepsilon\}\} \right) + \mathcal{O}(\phi^3). \tag{90}$$

In terms of an action in the parameter ϕ, the ϕ^0 term is a constant shift and will be neglected. For the linear term, note that

$$\{\phi, \varepsilon\} = \nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}}\varepsilon - \nabla_{\mathbf{p}}\phi \cdot \cancel{\nabla_{\mathbf{x}}\varepsilon} \equiv \varepsilon'(|\mathbf{p}|)\hat{n} \cdot \nabla_{\mathbf{x}}\phi, \tag{91}$$

where I noted that since $\varepsilon = \varepsilon(|\mathbf{p}|)$, its gradient is radial. When combined with the factor of f_0, the linear term is a total derivative in space and can be dropped. The result (91) can be substituted into the remaining quadratic term, giving

$$\begin{aligned} S_H &= -\frac{1}{2}\int dt \frac{d^2x d^2p}{(2\pi)^2} f_0 \{\phi, \varepsilon'(|\mathbf{p}|)\hat{n} \cdot \nabla_{\mathbf{x}}\phi\} + \mathcal{O}(\phi^3) \\ &= -\frac{1}{2}\int dt \frac{d^2x d^2p}{(2\pi)^2} f_0 \left[\nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}} \left(\varepsilon'(|\mathbf{p}|)\hat{n} \cdot \nabla_{\mathbf{x}}\phi \right) - \nabla_{\mathbf{p}}\phi \cdot \nabla_{\mathbf{x}} \left(\varepsilon'(|\mathbf{p}|)\hat{n} \cdot \nabla_{\mathbf{x}}\phi \right) \right] + \mathcal{O}(\phi^3) \\ &= \frac{1}{2}\int dt \frac{d^2x d^2p}{(2\pi)^2} \left[\varepsilon'(|\mathbf{p}|)\hat{n} \cdot \nabla_{\mathbf{x}}\phi \nabla_{\mathbf{p}} \cdot (f_0 \nabla_{\mathbf{x}}\phi) + f_0 \varepsilon'(|\mathbf{p}|) \nabla_{\mathbf{p}}\phi \cdot \nabla_{\mathbf{x}} \left(\hat{n} \cdot \nabla_{\mathbf{x}}\phi \right) \right] + \mathcal{O}(\phi^3), \end{aligned} \tag{92}$$

where the third equality is a result of IBP over momentum on the first term, as well as pulling out the factor of $\varepsilon'(|\mathbf{p}|)$ in the second term. Expanding the momentum derivative in the first term and replacing with (88),

$$\begin{aligned} S_H = \frac{1}{2}\int dt \frac{d^2x d^2p}{(2\pi)^2} \Big[&-\varepsilon'(|\mathbf{p}|)\delta(|\mathbf{p}| - p_F)(\hat{n} \cdot \nabla_{\mathbf{x}}\phi)^2 + \cancel{f_0 \varepsilon'(|\mathbf{p}|)(\hat{n} \cdot \nabla_{\mathbf{x}}\phi)\nabla_{\mathbf{p}} \cdot \nabla_{\mathbf{x}}\phi} \\ &+ \cancel{f_0 \varepsilon'(|\mathbf{p}|)\nabla_{\mathbf{p}}\phi \cdot \nabla_{\mathbf{x}}(\hat{n} \cdot \nabla_{\mathbf{x}}\phi)} \Big] + \mathcal{O}(\phi^3) \\ = -\frac{p_F}{2}\int dt \frac{d^2x d\theta}{(2\pi)^2} &\varepsilon'(p_F)(\hat{n} \cdot \nabla_{\mathbf{x}}\phi)^2 + \mathcal{O}(\phi^3), \end{aligned} \tag{93}$$

where the second and third terms canceled by swapping $\nabla_{\mathbf{p}} \cdot \nabla_{\mathbf{x}} \phi = \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{p}} \phi$ in the second then performing an IBP over space.

Combining both terms, the bosonized action (76) to Gaussian order in the bosonic parameter ϕ is

$$\boxed{S = -\frac{p_F}{2} \int dt \frac{d^2 x d\theta}{(2\pi)^2} (\hat{n} \cdot \nabla_{\mathbf{x}} \phi) \left(\dot{\phi} + v_F \hat{n} \cdot \nabla_{\mathbf{x}} \phi \right) + \mathcal{O}(\phi^3)} \tag{94}$$

in the semi-classical limit, where I noted $v_F = \varepsilon'(p_F)$ when $\varepsilon(|\mathbf{p}|) = |\mathbf{p}|^2/2m$. This is precisely the leading order Gaussian action quoted in [3, 8], albeit without Landau parameter interactions. The momentum space 2-point function $\langle \phi \phi' \rangle (\omega, \mathbf{q})$ follows naturally from this action. Note that $\partial_t \to \omega$, and $\nabla_{\mathbf{x}} \to -\mathbf{q}$ under a FT, giving

$$S = -\frac{i p_F}{2} \int d\tau \frac{d^2 q d\theta}{(2\pi)^2} \, \phi \left[(\hat{n} \cdot \mathbf{q}) \left(\omega - v_F \hat{n} \cdot \mathbf{q} \right) \right] \phi = \frac{1}{2} \int d\tau d^2 q d\theta \, \phi \left[\frac{-i p_F}{(2\pi)^2} (\hat{n} \cdot \mathbf{q}) \left(\omega - v_F \hat{n} \cdot \mathbf{q} \right) \right] \phi, \tag{95}$$

where inverting the kinetic operator gives [3]

$$\langle \phi \phi' \rangle (\omega, \mathbf{q}) = \frac{i(2\pi)^2}{p_F} \frac{\delta(\theta - \theta')}{\hat{n} \cdot \mathbf{q} (\omega - v_F \hat{n} \cdot \mathbf{q})}, \tag{96}$$

which can be used in diagrammatic expansions.

4 Coupling to Gauge Fields

4.1 Gauge Invariant Action

In this section, I will ensure local U(1) gauge invariance of the action (79)

$$S = \int dt \left\langle f_0, U^{-1} \star (i\partial_t - \varepsilon\star)U \right\rangle . \tag{97}$$

This will be done by noting that group action under U(1) shifts $U \to U' = W \star U = \exp(i\lambda(\mathbf{x}, t)) \star U$, where $W = \exp(i\lambda(\mathbf{x}, t))$ depends only on spacetime. If $\lambda = \lambda(t)$, the transformation would be global. As I prove below, gauge invariance is upheld by the minimal coupling procedure

$$i\partial_t \to i\partial_t - eA_0\star, \qquad\qquad \varepsilon(\mathbf{p}) \to \varepsilon(\mathbf{p} + e\mathbf{A}), \tag{98}$$

in which case the gauge invariant action is

$$S = \int dt \left\langle f_0, U^{-1} \star (i\partial_t - eA_0\star)U \right\rangle - \int dt \left\langle f_0, U^{-1} \star \varepsilon(\mathbf{p} + e\mathbf{A}) \star U \right\rangle , \tag{99}$$

and the gauge field A_μ must transform through

$$W^{-1} \star A_\mu \star W = A_\mu - \frac{1}{e}\partial_\mu \lambda. \tag{100}$$

Note the definition $\partial_\mu = (\partial_t, \nabla_\mathbf{x})$. There really isn't anything novel with this standard approach, aside from elevating matrix products to Moyal star-products. The logical next step is to pick a suitable gauge, which I will do in the next section.

Proof of Gauge Invariance: Let $U \to W \star U$ in minimally coupled action (99),

$$\begin{aligned} S &= \int dt \left\langle f_0, U^{-1} \star W^{-1} \star (i\partial_t - eA_0\star)W \star U \right\rangle - \int dt \left\langle f_0, U^{-1} \star W^{-1} \star \varepsilon(\mathbf{p} + e\mathbf{A}) \star W \star U \right\rangle \\ &= \int dt \left\langle f_0, U^{-1} \star i\partial_t U \right\rangle + \int dt \left\langle f_0, U^{-1} \star (W^{-1} \star i\partial_t W - eW^{-1} \star A_0 \star W) \star U \right\rangle \\ &\qquad - \int dt \left\langle f_0, U^{-1} \star W^{-1} \star \varepsilon(\mathbf{p} + e\mathbf{A}) \star W \star U \right\rangle . \end{aligned} \tag{101}$$

Note that, to leading order in λ,

$$W^{-1} \star i\partial_t W = \exp(-i\lambda) \star i\partial_t \exp(i\lambda) = i\partial_t(i\lambda), \qquad\qquad W^{-1} \star A_0 \star W = A_0 - \frac{1}{e}\partial_t \lambda, \tag{102}$$

where the second relation is a consequence of the choice in transformation (100). Putting these two together, I find that

$$W^{-1} \star i\partial_t W - eW^{-1} \star A_0 \star W = -\partial_t \lambda - eA_0 - e\left(-\frac{1}{e}\partial_t \lambda\right) = -eA_0, \tag{103}$$

and the second integral of (101) can be combined with the first. For the dispersion term, I have

$$\begin{aligned} W^{-1} \star \varepsilon(\mathbf{p} + e\mathbf{A}) \star W = \exp(-i\lambda) \star \varepsilon(\mathbf{p} + e\mathbf{A}) \star \exp(i\lambda) &= \varepsilon\left(\mathbf{p} + \nabla_\mathbf{x}\lambda + e\mathbf{A} - e\left(\frac{1}{e}\nabla_\mathbf{x}\lambda\right)\right) \\ &= \varepsilon(\mathbf{p} + e\mathbf{A}), \end{aligned} \tag{104}$$

where under the transformation I noted

$$
\begin{aligned}
\exp(-i\lambda) \star f(\mathbf{p}) \star \exp(i\lambda) = f(\mathbf{p}) + [-i\lambda, f(\mathbf{p})]_\star + \mathcal{O}(\lambda^2) &= f(\mathbf{p}) + \{\lambda, f(\mathbf{p})\}_M + \mathcal{O}(\lambda^2) \\
&\xrightarrow{\hbar\to 0} f(\mathbf{p}) + \nabla_{\mathbf{x}}\lambda \cdot \nabla_{\mathbf{p}} f(\mathbf{p}) + \mathcal{O}(\lambda^2) \\
&= f(\mathbf{p} + \nabla_{\mathbf{x}}\lambda),
\end{aligned} \tag{105}
$$

with the extra term being cancelled by the shift in $\mathbf{A}$ (100). Inputting these results into (101) makes it clear that S is U(1) gauge invariant.

4.2 Gauge Choice / New Coordinates

I now utilize the gauge freedom and choose the most convenient gauge for studies of magnetic fields in quantum systems - the symmetric gauge.

$$
A_0 = 0, \qquad \mathbf{A} = -\frac{B}{2}\mathbf{x} \times \hat{z}, \tag{106}
$$

where $\mathbf{x}$ is NOT the $\hat{x}$ direction, but rather the spatial coordinate. This gauge preserves the rotational symmetry about the $\mathbf{B}$-field direction, and allows me to choose a different basis for the phase space coordinates,

$$
\mathbf{R} = l_B^2(\mathbf{p} - e\mathbf{A}) \times \hat{z}, \qquad \mathbf{k} = \mathbf{p} + e\mathbf{A}, \tag{107}
$$

where $l_B^2 = 1/eB$ is the magnetic length, $\mathbf{R}$ is the guiding center coordinate (like the center of the cyclotron orbit), and $\mathbf{k}$ is simply the mechanical momentum (encodes the cyclotron motion). Note that $\mathbf{R}$ is not explicit in the gauge invariant action (101), which will be important later in the analysis of the action. The coordinates are defined this way by nature of their Moyal brackets,

$$
\{R_i, R_j\}_M = \varepsilon_{ij} l_B^2, \qquad \{k_i, k_j\} = -\varepsilon_{ij} l_B^{-2}, \qquad \{R_i, k_j\} = 0, \tag{108}
$$

which are still canonical.

Most importantly, the $(\mathbf{R}, \mathbf{k})$ basis uncouples the Moyal star product,

$$
F \star G = F \exp\left(\frac{i}{2B}(\overleftarrow{\partial}_{\mathbf{R}} \times \overrightarrow{\partial}_{\mathbf{R}})\right) \otimes \exp\left(-\frac{iB}{2}(\overleftarrow{\partial}_{\mathbf{k}} \times \overrightarrow{\partial}_{\mathbf{k}})\right) G \equiv F \star_{\mathbf{R}} \star_{\mathbf{k}} G, \tag{109}
$$

without any cross terms between $\mathbf{k}$ and $\mathbf{R}$, which I prove below.

Proof that Moyal Bracket Uncouples in New Coordinates (109): Since $\mathbf{A} = -\frac{B}{2}\mathbf{x} \times \hat{z}$, then I can write

$$
\mathbf{R} = \frac{1}{2}(\mathbf{x} \times \hat{z}) \times \hat{z} + l_B^2 \mathbf{p} \times \hat{z} = -\frac{\mathbf{x}}{2} + l_B^2 \mathbf{p} \times \hat{z}, \qquad \mathbf{k} = \mathbf{p} - \frac{1}{2l_B^2}(\mathbf{x} \times \hat{z}), \tag{110}
$$

where I used the fact that $(\mathbf{x} \times \hat{z}) \times \hat{z} = -\mathbf{x}$ when $\mathbf{x} \cdot \hat{z} = 0$. From the chain rule, I get that

$$
\begin{aligned}
\partial_{\mathbf{x}} &= \partial_{\mathbf{R}} \cdot \frac{\partial \mathbf{R}}{\partial \mathbf{x}} + \partial_{\mathbf{k}} \cdot \frac{\partial \mathbf{k}}{\partial \mathbf{x}} = -\frac{1}{2}\partial_{\mathbf{R}} + \frac{1}{2l_B^2}(\partial_{\mathbf{k}} \times \hat{z}), \\
\partial_{\mathbf{p}} &= \partial_{\mathbf{R}} \cdot \frac{\partial \mathbf{R}}{\partial \mathbf{p}} + \partial_{\mathbf{k}} \cdot \frac{\partial \mathbf{k}}{\partial \mathbf{p}} = -l_B^2(\partial_{\mathbf{R}} \times \hat{z}) + \partial_{\mathbf{k}},
\end{aligned} \tag{111}
$$

and

$$
\begin{aligned}
\overleftarrow{\partial}_{\mathbf{x}} \cdot \overrightarrow{\partial}_{\mathbf{p}} &= \left(-\frac{1}{2}\overleftarrow{\partial}_{\mathbf{R}} + \frac{1}{2l_B^2}(\overleftarrow{\partial}_{\mathbf{k}} \times \hat{z})\right) \cdot \left(-l_B^2(\overrightarrow{\partial}_{\mathbf{R}} \times \hat{z}) + \overrightarrow{\partial}_{\mathbf{k}}\right) \\
&= \frac{l_B^2}{2}\overleftarrow{\partial}_{\mathbf{R}} \cdot (\overrightarrow{\partial}_{\mathbf{R}} \times \hat{z}) - \frac{1}{2}\overleftarrow{\partial}_{\mathbf{R}} \cdot \overrightarrow{\partial}_{\mathbf{k}} - \frac{1}{2}(\overleftarrow{\partial}_{\mathbf{k}} \times \hat{z}) \cdot (\overrightarrow{\partial}_{\mathbf{R}} \times \hat{z}) + \frac{1}{2l_B^2}(\overleftarrow{\partial}_{\mathbf{k}} \times \hat{z}) \cdot \overrightarrow{\partial}_{\mathbf{k}} \\
&= \frac{l_B^2}{2}\overleftarrow{\partial}_{\mathbf{R}} \cdot (\overrightarrow{\partial}_{\mathbf{R}} \times \hat{z}) - \frac{1}{2}\overleftarrow{\partial}_{\mathbf{R}} \cdot \overrightarrow{\partial}_{\mathbf{k}} - \frac{1}{2}\overleftarrow{\partial}_{\mathbf{k}} \cdot \overrightarrow{\partial}_{\mathbf{R}} + \frac{1}{2l_B^2}(\overleftarrow{\partial}_{\mathbf{k}} \times \hat{z}) \cdot \overrightarrow{\partial}_{\mathbf{k}}, \\
\overleftarrow{\partial}_{\mathbf{p}} \cdot \overrightarrow{\partial}_{\mathbf{x}} &= \left(-l_B^2(\overleftarrow{\partial}_{\mathbf{R}} \times \hat{z}) + \overleftarrow{\partial}_{\mathbf{k}}\right) \cdot \left(-\frac{1}{2}\overrightarrow{\partial}_{\mathbf{R}} + \frac{1}{2l_B^2}(\overrightarrow{\partial}_{\mathbf{k}} \times \hat{z})\right) \\
&= \frac{l_B^2}{2}(\overleftarrow{\partial}_{\mathbf{R}} \times \hat{z}) \cdot \overrightarrow{\partial}_{\mathbf{R}} - \frac{1}{2}\overleftarrow{\partial}_{\mathbf{k}} \cdot \overrightarrow{\partial}_{\mathbf{R}} - \frac{1}{2}(\overleftarrow{\partial}_{\mathbf{R}} \times \hat{z}) \cdot (\overrightarrow{\partial}_{\mathbf{k}} \times \hat{z}) + \frac{1}{2l_B^2}\overleftarrow{\partial}_{\mathbf{k}} \cdot (\overrightarrow{\partial}_{\mathbf{k}} \times \hat{z}) \\
&= \frac{l_B^2}{2}(\overleftarrow{\partial}_{\mathbf{R}} \times \hat{z}) \cdot \overrightarrow{\partial}_{\mathbf{R}} - \frac{1}{2}\overleftarrow{\partial}_{\mathbf{k}} \cdot \overrightarrow{\partial}_{\mathbf{R}} - \frac{1}{2}\overleftarrow{\partial}_{\mathbf{R}} \cdot \overrightarrow{\partial}_{\mathbf{k}} + \frac{1}{2l_B^2}\overleftarrow{\partial}_{\mathbf{k}} \cdot (\overrightarrow{\partial}_{\mathbf{k}} \times \hat{z}).
\end{aligned}
\tag{112}
$$

where I used the vector identity $(\mathbf{A} \times \hat{z}) \cdot (\mathbf{B} \times \hat{z}) = \mathbf{A} \cdot \mathbf{B}$. When these two are subtracted, all cross terms between $\mathbf{k}, \mathbf{R}$ cancel, leaving just

$$
\begin{aligned}
\overleftarrow{\partial}_{\mathbf{x}} \cdot \overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}} \cdot \overrightarrow{\partial}_{\mathbf{x}} &= \frac{l_B^2}{2}\left(\overleftarrow{\partial}_{\mathbf{R}} \cdot (\overrightarrow{\partial}_{\mathbf{R}} \times \hat{z}) - (\overleftarrow{\partial}_{\mathbf{R}} \times \hat{z}) \cdot \overrightarrow{\partial}_{\mathbf{R}}\right) + \frac{1}{2l_B^2}\left((\overleftarrow{\partial}_{\mathbf{k}} \times \hat{z}) \cdot \overrightarrow{\partial}_{\mathbf{k}} - \overleftarrow{\partial}_{\mathbf{k}} \cdot (\overrightarrow{\partial}_{\mathbf{k}} \times \hat{z})\right) \\
&= \frac{1}{B}\overleftarrow{\partial}_{\mathbf{R}} \cdot (\overrightarrow{\partial}_{\mathbf{R}} \times \hat{z}) - B\overleftarrow{\partial}_{\mathbf{k}} \cdot (\overrightarrow{\partial}_{\mathbf{k}} \times \hat{z}) \\
&= \hat{z} \cdot \left[\frac{1}{B}\overleftarrow{\partial}_{\mathbf{R}} \times \overrightarrow{\partial}_{\mathbf{R}} - B\overleftarrow{\partial}_{\mathbf{k}} \times \overrightarrow{\partial}_{\mathbf{k}}\right],
\end{aligned}
\tag{113}
$$

where I replaced $l_B^2 = 1/B$ (for $e = 1$), and used the triple-product identity to rearrange things. The Moyal star product can now be expressed as

$$
\begin{aligned}
F \star G = F \exp\left(\frac{i}{2}\left(\overleftarrow{\partial}_{\mathbf{x}} \cdot \overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}} \cdot \overrightarrow{\partial}_{\mathbf{x}}\right)\right) G &= F \exp\left(\frac{i\hat{z}}{2} \cdot \left[\frac{1}{B}\overleftarrow{\partial}_{\mathbf{R}} \times \overrightarrow{\partial}_{\mathbf{R}} - B\overleftarrow{\partial}_{\mathbf{k}} \times \overrightarrow{\partial}_{\mathbf{k}}\right]\right) G \\
&= F \exp\left(\frac{i}{2B}\overleftarrow{\partial}_{\mathbf{R}} \times \overrightarrow{\partial}_{\mathbf{R}}\right) \otimes \exp\left(-\frac{iB}{2}\overleftarrow{\partial}_{\mathbf{k}} \times \overrightarrow{\partial}_{\mathbf{k}}\right) G, \qquad \square
\end{aligned}
\tag{114}
$$

where the separation of the exponential is valid because $\partial_{\mathbf{k}}$ and $\partial_{\mathbf{R}}$ commute.

We can now rewrite phase space functions as $F(\mathbf{x}, \mathbf{p}, t) \to F(\mathbf{R}, \mathbf{k}, t)$, and the inner product between them is becomes

$$
\langle f, F \rangle = \int \frac{d^2k d^2R}{(2\pi)^2} f(\mathbf{R}, \mathbf{k}) F(\mathbf{R}, \mathbf{k}), \tag{115}
$$

in the same way as the original phase space variables $(\mathbf{x}, \mathbf{p})$. The gauge invariant action (99) can be written in $(\mathbf{R}, \mathbf{k})$ basis as

$$
S = \int dt \left(\left\langle f_0, iU^{-1} \star_{\mathbf{R}} \star_{\mathbf{k}} \partial_t U \right\rangle - \langle f, \varepsilon \rangle\right), \tag{116}
$$

where f_0, f, U are all functions of $(\mathbf{R}, \mathbf{k})$, and $f = U \star_{\mathbf{R}} \star_{\mathbf{k}} f_0 \star_{\mathbf{R}} \star_{\mathbf{k}} U^{-1}$. Note that $\varepsilon = \varepsilon(\mathbf{k})$ only.

4.3 Dealing with Star Products / Noncommutative Representation

In order to make analytic progress, we need a way of truncating (or dealing with) the star products $\star_{\mathbf{R}}, \star_{\mathbf{k}}$. Note from their definitions (109) that the justification of truncating either product depends on the magnitude of the field B, which we are assuming is weak. This is no issue for $\star_{\mathbf{k}}$, as

$$B\overleftarrow{\partial}_{\mathbf{k}} \times \overrightarrow{\partial}_{\mathbf{k}} \sim \frac{B}{k_F^2} \sim \frac{\omega_c}{\mu} \ll 1. \tag{117}$$

since we are dealing with low energy fluctuations to the FS in weak magnetic fields. For $\star_{\mathbf{R}}$, however, ff the characteristic length scale of spatial fluctuations is L, then the argument to $\star_{\mathbf{R}}$ is

$$\frac{1}{B}\overleftarrow{\partial}_{\mathbf{r}} \times \overrightarrow{\partial}_{\mathbf{r}} \sim \frac{1/B}{L^2} \sim 1 \tag{118}$$

in the low energy (long wavelength) limit at low field. This means we need to treat $\star_{\mathbf{R}}$ differently. The trick in treating $\star_{\mathbf{R}}$ is to take a page out of the study of noncommutative field theory by making the replacement of the local guiding center coordinates $\mathbf{R}$ by Hermitian operators $\widehat{\mathbf{R}}$ which obey commutation relations consistent with the Moyal bracket of coordinates $\mathbf{R}$,

$$[\widehat{R}_i, \widehat{R}_j] = i\varepsilon_{ij} l_B^2. \tag{119}$$

This is done in practice by introducing the Weyl symbol (or Weyl transformation) for some general phase space function $F(\mathbf{R})$ [10]. If the Fourier conjugate $F(\mathbf{q})$ to $F(\mathbf{R})$ is given through

$$F(\mathbf{q}) = \int \frac{d^2R}{2\pi l_B^2} F(\mathbf{R}) e^{i\mathbf{q}\cdot\mathbf{R}}, \tag{120}$$

then the associated Weyl transform is

$$F^W(\widehat{\mathbf{R}}) = \int \frac{d^2q}{2\pi l_B^{-2}} F(\mathbf{q}) e^{-i\mathbf{q}\cdot\widehat{\mathbf{R}}} = \int \frac{d^2Rd^2q}{(2\pi)^2} F(\mathbf{R}) e^{i\mathbf{q}\cdot(\mathbf{R}-\widehat{\mathbf{R}})}. \tag{121}$$

Note how I redefined the Fourier transform slightly, separating the factor of $(2\pi)^2$ onto both the Fourier and inverse-Fourier transforms, and introducing factors of the magnetic length. As I will show, factors of l_B^2 are convenient since they appear in commutation relations for the $\widehat{\mathbf{R}}$ variables.[8]

Now, why is it helpful in dealing with $\star_{\mathbf{R}}$ to work on a noncommutative space? This is through the relation

$$\int d^2R\, F_1(\mathbf{R}) \star_{\mathbf{R}} F_2(\mathbf{R}) \star_{\mathbf{R}} \cdots \star_{\mathbf{R}} F_n(\mathbf{R}) = 2\pi l_B^2 \mathrm{tr}\left[F_1^W(\hat{\mathbf{R}}) F_2^W(\hat{\mathbf{R}}) \cdots F_n^W(\hat{\mathbf{R}})\right], \tag{122}$$

which I elaborate on below.

Idea Behind Relation (122): For two functions,

$$\mathrm{tr}[F_{1,\hat{\mathbf{R}}} F_{2,\hat{\mathbf{R}}}] = \int \frac{d^2R_1 d^2q_1 d^2R_2 d^2q_2}{(2\pi)^4} F_1(\mathbf{R}_1) F_2(\mathbf{R}_2) e^{i\mathbf{q}_1\cdot(\mathbf{R}_1-\widehat{\mathbf{R}})} e^{i\mathbf{q}_2\cdot(\mathbf{R}_2-\widehat{\mathbf{R}})}. \tag{123}$$

Note from BCH formula (which truncates naturally to leading order) that

$$e^{-i\mathbf{q}_1\cdot\widehat{\mathbf{R}}} e^{-i\mathbf{q}_2\cdot\widehat{\mathbf{R}}} = \exp\left(-i(\mathbf{q}_1+\mathbf{q}_2)\cdot\widehat{\mathbf{R}} - \frac{il_B^2}{2}\mathbf{q}_1\times\mathbf{q}_2\right). \tag{124}$$

Then perform the trace over $\widehat{\mathbf{R}}$, or integrating over them as coordinates, getting $\delta(\mathbf{q}_1+\mathbf{q}_2)$. Then, the other term is simplified by replacing $\mathbf{q}_i \to -i\nabla_{\mathbf{R}_i}$ and is practically $\star_{\mathbf{R}}$. After using the above delta function, do the remaining integration over $\mathbf{q}$ to get $\delta(\mathbf{R}_1-\mathbf{R}_2)$ and the answer falls out. The $2\pi l_B^2$ pre-factor is a consequence of the convention (120).

[8] This is the same reason for why in typical stat-mech, $[x,p] \propto \hbar$ results in factors of $1/\hbar$ in the phase-space differential.

It is also helpful because the $\mathbf{R}$ coordinate does not appear explicitly in the gauge invariant action, a consequence of magnetic translation symmetry which I will discuss later. I will first rewrite (116) by expanding out the inner product,

$$S = \int dt \frac{d^2k d^2R}{(2\pi)^2} \left(f_0(\mathbf{R},\mathbf{k}) i U^{-1}(\mathbf{R},\mathbf{k}) \star_{\mathbf{R}} \star_{\mathbf{k}} \partial_t U(\mathbf{R},\mathbf{k}) - f(\mathbf{R},\mathbf{k})\varepsilon(\mathbf{k}) \right). \tag{125}$$

In order to apply (122), the entire integrand should be written as consecutive $\star_{\mathbf{R}}$ products, and for that I will appeal to

$$\int d^2R\, F(\mathbf{R}) G(\mathbf{R}) = \int d^2R\, F(\mathbf{R}) \star_{\mathbf{R}} G(\mathbf{R}), \tag{126}$$

which I prove below.

Proof of the Integral Relation (126): From the definition of $\star_{\mathbf{R}}$,

$$\begin{aligned}\int d^2R\, F(\mathbf{R}) \star_{\mathbf{R}} G(\mathbf{R}) &= \int d^2R\, F(\mathbf{R}) \exp\left(\frac{i l_B^2}{2} \overleftarrow{\partial}_{\mathbf{R}} \times \overrightarrow{\partial}_{\mathbf{R}} \right) G(\mathbf{R}) \\ &= \int d^2R \frac{d^2q d^2q'}{(2\pi l_B^{-2})^2} \widetilde{F}(\mathbf{q}) \widetilde{G}(\mathbf{q}') \exp\left(-i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{R} - \frac{i l_B^2}{2} \mathbf{q}\times\mathbf{q}' \right),\end{aligned} \tag{127}$$

where I replaced $\overleftarrow{\partial}_{\mathbf{R}} \to -i\mathbf{q}$ and $\overrightarrow{\partial}_{\mathbf{R}} \to -i\mathbf{q}'$ upon substituting the Fourier transforms. The integration over $\mathbf{R}$ can be performed, giving a factor of $(2\pi)^2\delta(\mathbf{q}+\mathbf{q}')$ that results in

$$\int d^2R\, F(\mathbf{R}) \star_{\mathbf{R}} G(\mathbf{R}) = \int \frac{d^2q}{l_B^{-4}} \widetilde{F}(\mathbf{q}) \widetilde{G}(-\mathbf{q}) \exp\left(\frac{i l_B^2}{2} \mathbf{q}\times\mathbf{q} \right) = \int \frac{d^2q}{l_B^{-4}} \widetilde{F}(\mathbf{q}) \widetilde{G}(-\mathbf{q}). \tag{128}$$

Fourier transforming back, I find

$$\begin{aligned}\int d^2R\, F(\mathbf{R}) \star_{\mathbf{R}} G(\mathbf{R}) = \int \frac{d^2R d^2R'}{(2\pi l_B^2)^2} \frac{d^2q}{l_B^{-4}} F(\mathbf{R}) G(\mathbf{R}') e^{i\mathbf{q}\cdot(\mathbf{R}-\mathbf{R}')} &= \int \frac{d^2R d^2R'}{(2\pi)^2} F(\mathbf{R}) G(\mathbf{R}') (2\pi)^2 \delta(\mathbf{R}-\mathbf{R}') \\ &= \int d^2R\, F(\mathbf{R}) G(\mathbf{R}),\end{aligned} \tag{129}$$

as desired.

This allows me to write

$$\begin{aligned} S &= \int dt \frac{d^2k d^2R}{(2\pi)^2} \left(f_0(\mathbf{R},\mathbf{k}) \star_{\mathbf{R}} iU^{-1}(\mathbf{R},\mathbf{k}) \star_{\mathbf{R}} \star_{\mathbf{k}} \partial_t U(\mathbf{R},\mathbf{k}) - f(\mathbf{R},\mathbf{k}) \star_{\mathbf{R}} \varepsilon(\mathbf{k}) \right) \\ &= \int dt \frac{d^2k}{2\pi l_B^{-2}} \operatorname{tr}\left[f_{0,\widehat{\mathbf{R}}}(\mathbf{k}) U_{\widehat{\mathbf{R}}}^{-1} \star_{\mathbf{k}} i\partial_t U_{\widehat{\mathbf{R}}} - f_{\widehat{\mathbf{R}}}(\mathbf{k})\varepsilon(\mathbf{k}) \right],\end{aligned} \tag{130}$$

where I redefined the notation for the Weyl transform of $F(\mathbf{R})$ as $F_{\widehat{\mathbf{R}}}$, for brevity and to align with the work [14]. Now that the integration over $\mathbf{R}$ and star product $\star_{\mathbf{R}}$ is buried in trace over operators, I will redefine in terms of the $\mathbf{k}$ subspace. Let the inner product $\langle f, G\rangle_{\mathbf{k}}$ be defined through

$$\langle f(\mathbf{k}), G(\mathbf{k})\rangle_{\mathbf{k}} \equiv \int \frac{d^2k}{2\pi l_B^{-2}} f(\mathbf{k}) G(\mathbf{k}), \tag{131}$$

giving the simple form

$$S = \int dt \operatorname{tr}\left[\left\langle f_{0,\widehat{\mathbf{R}}}, U_{\widehat{\mathbf{R}}}^{-1} \star_{\mathbf{k}} i\partial_t U_{\widehat{\mathbf{R}}} \right\rangle - \left\langle f_{\widehat{\mathbf{R}}}, \varepsilon(\mathbf{k}) \right\rangle \right], \tag{132}$$

where I made the dependence of ε on $\mathbf{k}$ explicit to highlight the fact that it is independent of $\widehat{\mathbf{R}}$.

To be perfectly clear, the dynamic operator is $U_{\widehat{\mathbf{R}}} = \exp\left(i\phi_{\widehat{\mathbf{R}}}(\mathbf{k},t)\right)$, and the orbit of the one-particle distribution function of the ground state is

$$
\begin{aligned}
f_{\widehat{\mathbf{R}}}(\mathbf{k},t) = U_{\widehat{\mathbf{R}}} \star_{\mathbf{k}} f_{0,\widehat{\mathbf{R}}}(\mathbf{k}) \star_{\mathbf{k}} U_{\widehat{\mathbf{R}}}^{-1} = \exp\left(i\phi_{\widehat{\mathbf{R}}}(\mathbf{k},t)\right) \star_{\mathbf{k}} f_{0,\widehat{\mathbf{R}}} \star_{\mathbf{k}} \exp\left(-i\phi_{\widehat{\mathbf{R}}}(\mathbf{k},t)\right) &= f_{0,\widehat{\mathbf{R}}} + [i\phi_{\widehat{\mathbf{R}}}, f_{0,\widehat{\mathbf{R}}}]_{\star_{\mathbf{k}}} + \mathcal{O}(\phi^2) \\
&= f_{0,\widehat{\mathbf{R}}} - \{\phi_{\widehat{\mathbf{R}}}, f_{0,\widehat{\mathbf{R}}}\}_{\mathbf{k}} + \mathcal{O}(\phi^2),
\end{aligned}
\tag{133}
$$

also known as the one-particle density matrix for the coherent state. Here, the bracket $\{F,G\}_{\mathbf{k}}$ is defined through

$$
\begin{aligned}
\{F,G\}_{\mathbf{k}} &= \frac{1}{i}(F \star_{\mathbf{k}} G - G \star_{\mathbf{k}} F) \\
&= \frac{1}{i}\left(F \exp\left(-\frac{iB}{2}\overleftarrow{\partial}_{\mathbf{k}} \times \overrightarrow{\partial}_{\mathbf{k}}\right) G - G \exp\left(-\frac{iB}{2}\overleftarrow{\partial}_{\mathbf{k}} \times \overrightarrow{\partial}_{\mathbf{k}}\right) F\right),
\end{aligned}
\tag{134}
$$

which limits to the Poisson-like bracket

$$
\{F,G\}_{\mathbf{k}} \xrightarrow{\text{semi.}} \frac{1}{i}\left(\left(-\frac{iB}{2}\partial_{\mathbf{k}}F \times \partial_{\mathbf{k}}G\right) - \left(-\frac{iB}{2}\partial_{\mathbf{k}}G \times \partial_{\mathbf{k}}F\right)\right) = -B\,\partial_{\mathbf{k}}F \times \partial_{\mathbf{k}}G \tag{135}
$$

in the semi-classical limit.

4.4 Magnetic Translation Symmetry

I now need a way to perform the trace over noncommuting operators $\widehat{\mathbf{R}}$ in the action (132). To do this, I note that in the presence of a magnetic field, the original translation symmetry is broken, as the Bloch momentum is replaced with the proper mechanical momentum $\mathbf{k} = \mathbf{p} + e\mathbf{A}$. Now, I have a magnetic translation symmetry with an operator defined through [14]

$$
\widehat{t}(\mathbf{d}) = \exp\left(-\frac{i}{l_B^2}(d_x\widehat{R}_y - d_y\widehat{R}_x)\right). \tag{136}
$$

All $\widehat{t}(\mathbf{d})$'s commute with the Hamiltonian, allowing me to use the eigenbasis of $\widehat{t}(\mathbf{d})$ to evaluate the trace. I will first elaborate on some aspects of magnetic translation symmetry.

It's important to note that the $\widehat{t}(\mathbf{d})$ do not necessarily commute with themselves,

$$
\widehat{t}(\mathbf{l}_1)\widehat{t}(\mathbf{l}_2) = \exp\left(-\frac{i}{l_B^2}\left((l_x^1 + l_x^2)\widehat{R}_y - (l_y^1 + l_y^2)\widehat{R}_x\right) - \frac{i}{2l_B^2}(\mathbf{l}_1 \times \mathbf{l}_2)_z\right), = \exp\left(-\frac{i}{l_B^2}(\mathbf{l}_1 \times \mathbf{l}_2)_z\right)\widehat{t}(\mathbf{l}_2)\widehat{t}(\mathbf{l}_1), \tag{137}
$$

where I used the BCH formula. We want the symmetry group to be abelian, as that allows us to form simultaneous eigenstates between any two $\widehat{t}(\mathbf{l})$. This means the lattice translation symmetry is broken into a discrete subgroup defined by the condition

$$
|\mathbf{l}_1 \times \mathbf{l}_2| = 2\pi l_B^2 \iff B|\mathbf{l}_1 \times \mathbf{l}_2| = \Phi_0. \tag{138}
$$

Having chosen the smallest ($n = 1$) case (138), the vectors $\mathbf{l}_{1,2}$ define a magnetic unit cell which encloses a single magnetic flux quantum $\Phi_0 = 2\pi$, and the associated region in momentum space is known as the magnetic Brillouin zone, in which Bloch-like states can be defined [4].

I am free to assume that the two $\mathbf{l}_i$ live on a torus with lengths $L_i = M_i|\mathbf{l}_i|$, and $\mathbf{l}_1 \perp \mathbf{l}_2$ (since $\mathbf{k}$-space is periodic I can just wrap it into the 1mBZ). Any general linear combination of $\mathbf{l}_i$'s can be written as $\mathbf{l} = n_1\mathbf{l_1} + n_2\mathbf{l_2}$, where $n_i = 1, 2, \cdots, M_i$, which defines the set of magnetic translation operators $\mathbb{T} = \{\widehat{t}(\mathbf{l})\}$ for the torus. The Bloch states may then be written in the usual manner[9]

$$
|\mathbf{K}\rangle = e^{i\mathbf{K}\cdot\widehat{\mathbf{R}}}|\mathbf{0}\rangle, \qquad \mathbf{K} = \frac{2\pi n_1}{L_1}\hat{x} + \frac{2\pi n_2}{L_2}\hat{y}, \tag{140}
$$

[9]The zero momentum state satisfies [7]

$$
\widehat{t}(\mathbf{l}) = \xi(\mathbf{l})\,|\mathbf{0}\rangle, \tag{139}
$$

where $\xi(\mathbf{l}) = 1$ if $\widehat{t}(\mathbf{l}/2) \in \mathbb{T}$ and $\xi(\mathbf{l}) = -1$ otherwise. Choosing $\xi(\mathbf{l}) = -1$ for the case that $\widehat{t}(\mathbf{l}/2) \notin \mathbb{T}$ ensures that $\widehat{t}(\mathbf{l})\widehat{t}(\mathbf{l})$, gets a factor of $(-1)^2 = 1$, as it should since translating half of $\widehat{t}(\mathbf{l})\widehat{t}(\mathbf{l})$ is just translating by $\widehat{t}(\mathbf{l})$, which is an element of $\mathbb{T}$ by assumption.

where the $\mathbf{K}$ are Bloch momenta in the (first) mBZ, the total number of which being equal to the total flux quantum in the system (or total number of magnetic unit cells)

$$\#\text{ of Bloch momenta} = M_1 M_2 = \frac{B(M_1 M_2 |\mathbf{l}_1 \times \mathbf{l}_2|)}{2\pi} = \frac{BL_1 L_2}{\Phi_0} = N_\phi. \tag{141}$$

The $|\mathbf{k}\rangle$ are eigenstates of the magnetic translation operator $\widehat{t}(\mathbf{l})$,

$$\hat{t}(\mathbf{l})\,|\mathbf{K}\rangle = \xi(\mathbf{k}) e^{i\mathbf{K}\cdot\mathbf{l}}\,|\mathbf{K}\rangle\,, \tag{142}$$

which is easy to show by expanding $|\mathbf{K}\rangle$ in terms of the zero-momentum state, commuting operators through the BCH formula. They subsequently form an orthonormal and complete basis for the non-commutative manifold $\widehat{\mathbf{R}}$, and the trace over non-commutative coordinates can be replaced by

$$\mathrm{tr}[\cdots] = \sum_{\mathbf{K}\in\mathrm{mBZ}} \langle\mathbf{K}|\cdots|\mathbf{K}\rangle\,, \tag{143}$$

in the action (132), giving

$$S = \sum_{\mathbf{K}\in\mathrm{mBZ}} \int dt\ \langle\mathbf{K}|\left[\left\langle f_{0,\widehat{\mathbf{R}}}, U_{\widehat{\mathbf{R}}}^{-1} \star_{\mathbf{k}} i\partial_t U_{\widehat{\mathbf{R}}}\right\rangle - \left\langle f_{\widehat{\mathbf{R}}}, \varepsilon(\mathbf{k})\right\rangle\right]|\mathbf{K}\rangle\,. \tag{144}$$

Now, since the Hamiltonian has magnetic translation symmetry, and the distribution function $f_{\widehat{\mathbf{R}}}(\mathbf{k},t)$ is given from $f_{0,\widehat{\mathbf{R}}}(\mathbf{k})$ under Hamiltonian evolution, $f_{\widehat{\mathbf{R}}}(\mathbf{k},t)$ should also possess a magnetic translation symmetry. Without prescribing an explicit Hamiltonian, I can constrain the orbit to possessing this symmetry by

$$\widehat{t}(\mathbf{l}) f_{\widehat{\mathbf{R}}}(\mathbf{k},t)\widehat{t}^{-1}(\mathbf{l}) = f_{\widehat{\mathbf{R}}}(\mathbf{k},t), \tag{145}$$

which in turn places constraints on both $f_{0,\widehat{\mathbf{R}}}(\mathbf{k})$ and $\phi_{\widehat{\mathbf{R}}}(\mathbf{k},t)$. From (142),

$$\left\langle\mathbf{K}|f_{\widehat{\mathbf{R}}}|\mathbf{K}'\right\rangle = \left\langle\mathbf{K}|\widehat{t}(\mathbf{l}) f_{\widehat{\mathbf{R}}}\widehat{t}^{-1}(\mathbf{l})|\mathbf{K}'\right\rangle = e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{l}}\left\langle\mathbf{K}|f_{\widehat{\mathbf{R}}}|\mathbf{K}'\right\rangle \iff \left\langle\mathbf{K}|f_{\widehat{\mathbf{R}}}|\mathbf{K}'\right\rangle = \left\langle\mathbf{K}|f_{\widehat{\mathbf{R}}}|\mathbf{K}\right\rangle\delta_{\mathbf{K},\mathbf{K}'}. \tag{146}$$

since $\xi(\mathbf{k})^2 = 1$. From an expansion of (133), it follows that

$$\left\langle\mathbf{K}|f_{0,\widehat{\mathbf{R}}}(\mathbf{k},t)|\mathbf{K}'\right\rangle = \widetilde{f}_{0,\mathbf{K}}(\mathbf{k},t)\delta_{\mathbf{K},\mathbf{K}'}, \qquad\qquad \left\langle\mathbf{K}|U_{\widehat{\mathbf{R}}}(\mathbf{k},t)|\mathbf{K}'\right\rangle = \widetilde{U}_{\mathbf{K}}(\mathbf{k},t)\delta_{\mathbf{K},\mathbf{K}'}, \tag{147}$$

Here, $\widetilde{f}_{0,\mathbf{K}}(\mathbf{k},t)$ is simply the expectation value of $\widetilde{f}_{0,\widehat{R}}(\mathbf{k},t)$ in the Bloch state $|\mathbf{K}\rangle$, and similarly for $\widetilde{U}_{\mathbf{K}}$. So, as a matrix in the $\mathbf{K}$ basis, even though $\phi_{\widehat{\mathbf{R}}}$ has $N_\phi \times N_\phi$ independent components, magnetic translation symmetry constrains the number of relevant modes to be N_ϕ. If all were included, I would be over-counting the number of modes, and would derive an incorrect (non-extensive) free energy.

4.5 Effective Action / Windings

By inserting completeness relations into (144) and noting that all expectation values are diagonal by nature of the magnetic translation symmetry, I find

$$S = \sum_{\mathbf{K}\in\mathrm{mBZ}} \int dt\ \left[\left\langle f_0(\mathbf{k}), \widetilde{U}_{\mathbf{K}}^{-1} \star_{\mathbf{k}} i\partial_t \widetilde{U}_{\mathbf{K}}\right\rangle - \left\langle \widetilde{f}_{\mathbf{K}}, \varepsilon(\mathbf{k})\right\rangle\right], \tag{148}$$

where I used the fact that for a fixed chemical potential (size of the FS), $\widetilde{f}_0$ does not depend on the Bloch momentum $\mathbf{K}$ explicitly. It just depends on the dispersion $\varepsilon(\mathbf{k})$, which in the semiclassical limit is $f_0(\mathbf{k}) = \Theta(\mu - \varepsilon(\mathbf{k})) = \Theta(k_F - |\mathbf{k}|)$. Under the dynamic operator $\widetilde{U}_{\mathbf{K}} = \exp(i\widetilde{\phi}_{\mathbf{K}})$, the action (148) expands to third order in $\widetilde{\phi}_{\mathbf{K}}$ as

$$\begin{aligned} S = \sum_{\mathbf{K}\in\mathrm{mBZ}} \int dt \left\langle f_0, -\dot{\phi}_{\mathbf{K}} - \frac{1}{2}\{\phi_{\mathbf{K}}, \dot{\phi}_{\mathbf{K}}\} - \frac{1}{6}\{\phi_{\mathbf{K}}, \{\phi_{\mathbf{K}}, \dot{\phi}_{\mathbf{K}}\}\} + \cdots \right\rangle \\ + \sum_{\mathbf{K}\in\mathrm{mBZ}} \int dt \left\langle f_0 - \{\phi_{\mathbf{K}}, f_0\} + \frac{1}{2}\{\phi_{\mathbf{K}}, \{\phi_{\mathbf{K}}, f_0\}\} - \frac{1}{6}\{\phi_{\mathbf{K}}, \{\phi_{\mathbf{K}}, \{\phi_{\mathbf{K}}, f_0\}\}\} + \cdots, -\varepsilon(\mathbf{k}) \right\rangle \end{aligned} \tag{149}$$

where I replaced $\widetilde{\phi} \to \phi$ and dropped $\mathbf{k}$ subscripts on inner products and Poisson brackets. Note that $\widetilde{\phi}_{\mathbf{K}}$ is a compact boson (defined mod 2π):

$$\widetilde{\phi}_{\mathbf{K}}(\mathbf{k},t) = \widetilde{\phi}_{\mathbf{K}}(\mathbf{k},t) + 2\pi, \tag{150}$$

so we must be careful in capturing winding configurations in the action. This can be seen by an IBP on the following Poisson bracket,

$$\begin{aligned}
\langle f, \{F, \phi_{\mathbf{K}}\}\rangle = \int \frac{d^2k}{2\pi l_B^{-2}} f\{F,\phi_{\mathbf{K}}\} &= -B\varepsilon_{ij}\int \frac{d^2k}{2\pi l_B^{-2}} f\partial_{k_i}F\partial_{k_j}\phi_{\mathbf{K}} \\
&= B\varepsilon_{ij}\int \frac{d^2k}{2\pi l_B^{-2}}\partial_{k_i}\left(f\partial_{k_j}\phi_{\mathbf{K}}\right)F \qquad \text{(IBP)} \\
&= B\varepsilon_{ij}\int \frac{d^2k}{2\pi l_B^{-2}}(\partial_{k_i}f\partial_{k_j}\phi_{\mathbf{K}})F + B\varepsilon_{ij}\int \frac{d^2k}{2\pi l_B^{-2}} fF\partial_{k_i}\partial_{k_j}\phi_{\mathbf{K}} \\
&= -\int \frac{d^2k}{2\pi l_B^{-2}}\{f,\phi_{\mathbf{K}}\}F + \int \frac{d^2k}{2\pi l_B^{-2}} BfF\partial_{\mathbf{k}}\times\partial_{\mathbf{k}}\phi_{\mathbf{K}} \\
&= -\left\langle \{f,\phi_{\mathbf{K}}\}, F\right\rangle + B\left\langle f, F\partial_{\mathbf{k}}\times\partial_{\mathbf{k}}\phi_{\mathbf{K}}\right\rangle,
\end{aligned} \tag{151}$$

where the curl of a gradient is nonzero when the function it acts on is multi-valued (has winding configurations). This is precisely the same treatment taken in analyses of vortices of the XY model. I will now analyze each term individually, breaking apart winding terms from non-winding terms. This is done in Section 4.8, and the full action takes the form

$$\begin{aligned}
S_{\text{cb}} &= -\sum_{i=1}^{N_\phi}\int \frac{dtd\theta}{4\pi}\partial_\theta\phi_i\left[\dot{\phi}_i + \omega_c\partial_\theta\phi_i\right], \\
S_{\text{w}} &= -\sum_{i=1}^{N_\phi}\int dt\Big[\left\langle f_0, \dot{\phi}_i\right\rangle - \left\langle f_0, B(\partial_{\mathbf{k}}\times\partial_{\mathbf{k}}\phi_i)\left(\frac{1}{2}\dot{\phi}_i + \frac{1}{6}\{\phi_i,\dot{\phi}_i\} + \cdots\right)\right\rangle \\
&\qquad - \left\langle B(\partial_{\mathbf{k}}\times\partial_{\mathbf{k}}\phi_i)\left(\frac{1}{2}\{\phi_i, f_0\} - \frac{1}{6}\{\phi_i,\{\phi_i,f_0\}\}\right), \varepsilon(\mathbf{k})\right\rangle\Big]
\end{aligned} \tag{152}$$

where for notational purposes I noted that since there are N_ϕ different Bloch momenta $\mathbf{K}$ in the mBZ, we have N_ϕ bosons which can be instead labeled by some index $i \in [1, N_\phi]$. Here we have N_ϕ uncoupled and identical actions, and this degeneracy already sheds some light towards the presence of Landau level physics. I labeled the Floreanini-Jackiw action S_{cb} as such because it describes N_ϕ species of *chiral bosons* in momentum space with angular speed ω_c. In position space, S_{cb} describes the edge state of an integer quantum Hall state with Chern number N_ϕ, which is a similar picture to the semi-classical notion of electron's moving around the FS in the presence of a magnetic field. This implies that cyclotron motion can be viewed as edge states of a Chern insulator in momentum space.

To finish up this section, I will make a simplification of the winding action by imposing winding configurations of ϕ, which I take to be $\phi \sim \widetilde{p}\theta$. These configurations contribute to

$$\int d\theta\ \partial_\theta\phi = 2\pi\widetilde{p}. \tag{153}$$

In analogy to the XY model, I will define a path $\partial\Sigma$ which circles a $\mathbf{k}$-space vortex and defines an oriented area Σ . This allows me to write

$$2\pi\widetilde{p} = \int_{\partial\Sigma} d\theta\hat{\theta}\cdot\partial_{\mathbf{k}}\phi = \int_\Sigma d\mathbf{a}\cdot(\partial_{\mathbf{k}}\times\partial_{\mathbf{k}}\phi) \iff \partial_{\mathbf{k}}\times\partial_{\mathbf{k}}\phi = 2\pi\widetilde{p}\delta(\mathbf{k}). \tag{154}$$

In S_{w}, this has the consequence that all terms with Poisson brackets vanish, due to disjoint support, leaving just

$$
\begin{aligned}
S_{\rm cb} &= -\sum_{i=1}^{N_\phi} \int \frac{dt d\theta}{4\pi} \partial_\theta \phi_i \left[\dot{\phi}_i + \omega_c \partial_\theta \phi_i\right], \\
S_{\rm w} &= \sum_{i=1}^{N_\phi} \int \frac{dt d^2 k}{2\pi B} f_0(\mathbf{k}) \left(-\dot{\phi}_i + \frac{B}{2}(\partial_{\mathbf{k}} \times \partial_{\mathbf{k}} \phi_i)\dot{\phi}_i\right)
\end{aligned}
\tag{155}
$$

as the fully simplified action $S = S_{\rm cb} + S_{\rm w}$. The action $S_{\rm cb}$ is defined differently than [14], as they assume θ is measured clockwise.

4.6 Quantum Oscillation

Following the approach used in [13], I will derive the zero-mode action by taking the first two terms in the mode expansion of ϕ, which is just $\phi = q + p\theta$. This results in

$$
\begin{aligned}
S_{\rm zero} &= \int dt \left(-\frac{1}{2\pi B}\left[\int d^2k \; f_0(\mathbf{k})\right]\dot{q} + p\dot{q} - \frac{1}{2}\left[\int \frac{d\theta}{2\pi} \frac{B\kappa(\theta)|\partial_{\mathbf{k}}\varepsilon(\mathbf{k})|}{1 + B\Omega^{(0)}}\right] p^2\right) \\
&= \int dt \left(-\frac{\mathcal{A}}{2\pi B}\dot{q} + p\dot{q} - \frac{\omega_c}{2}p^2\right)
\end{aligned}
\tag{156}
$$

where I have defined the Fermi surface area $\mathcal{A}$. This is precisely the action for a particle moving on a ring with a unit radius ($q \sim q + 2\pi$), in the presence of a magnetic flux $\mathcal{A}/B$ penetrating the center of the ring [14]. By evaluating the path integral explicitly, one can show that the topological θ-term (proportional to $\dot{q}$) gives rise to an oscillatory component of the free energy with period (in $1/B$) of $2\pi/\mathcal{A}$.

4.7 Summary of the Gauge Coupling Section

- Introduce magnetic field by making the action locally U(1) gauge invariant.
- Most common choice is symmetric gauge, with coordinate basis $(\mathbf{R}, \mathbf{k})$. Conveniently, $\star = \star_{\mathbf{R}} \star_{\mathbf{k}}$ uncouples.
- To get effective semi-classical action, are justified in weak magnetic field to truncate $\star_{\mathbf{k}}$, but $\star_{\mathbf{R}}$ must be done differently. Action doesn't explicitly depend on $\mathbf{R}$ via magnetic translation symmetry, so we elevate $\mathbf{R} \to \widehat{\mathbf{R}}$ to non-commutative coordinates and choose the eigenbasis of the magnetic translation symmetry operator for the manifold of $\widehat{\mathbf{R}}$. This is sketched by
$$
\int \frac{d^2R}{2\pi B^{-1}}(\cdots) \mapsto \mathrm{tr}_{\widehat{\mathbf{R}}}[\cdots] \mapsto \sum_{\mathbf{K} \in \mathrm{mBZ}} \langle \mathbf{K}_i | \cdots | \mathbf{K}_i \rangle \, .
\tag{157}
$$
There are N_ϕ such $\mathbf{K}_i$ in the mBZ, where N_ϕ is the total number of magnetix flux quanta through the system (or L.L. degen.) Invariance under magnetic translation symmetry ensures that matrix elements of ϕ, f_0, f in the $\mathbf{K}$ basis are diagonal.
- Now truncate $\star_{\mathbf{k}}$. To get "proper" action, need to use IBP to get derivatives onto f_0. Since $\widetilde{\phi}$ is a compact variable, total derivatives cannot be dropped and constitute a winding term. The non-winding action describes N_ϕ flavors of chiral bosons with angular speed ω_c in momentum space.

"cyclotron motion $\iff$ edge states of momentum space Chern insulator"

4.8 Derivation of Chiral Boson and Winding Actions

WZW Term

Through a use of (151), the WZW term of (149) becomes

$$\begin{aligned}S_{\mathrm{WZW}} &= -\sum_{\mathbf{K}\in\mathrm{mBZ}}\int dt\left[\left\langle f_0,\dot{\phi}_{\mathbf{K}}\right\rangle - \frac{1}{2}\left\langle f_0,\{\dot{\phi}_{\mathbf{K}},\phi_{\mathbf{K}}\}\right\rangle - \frac{1}{6}\left\langle f_0,\{\{\phi_{\mathbf{K}},\dot{\phi}_{\mathbf{K}}\},\phi_{\mathbf{k}}\}\right\rangle+\cdots\right]\\ &= -\sum_{\mathbf{K}\in\mathrm{mBZ}}\int dt\left[\frac{1}{2}\left\langle\{f_0,\phi_{\mathbf{K}}\},\dot{\phi}_{\mathbf{K}}\right\rangle + \frac{1}{6}\left\langle\{f_0,\phi_{\mathbf{K}}\},\{\phi_{\mathbf{K}},\dot{\phi}_{\mathbf{K}}\}\right\rangle+\cdots\right]\\ &\quad -\sum_{\mathbf{K}\in\mathrm{mBZ}}\int dt\left[\left\langle f_0,\dot{\phi}_{\mathbf{K}}\right\rangle - \left\langle f_0, B(\partial_{\mathbf{k}}\times\partial_{\mathbf{k}}\phi_{\mathbf{K}})\left(\frac{1}{2}\dot{\phi}_{\mathbf{K}}+\frac{1}{6}\{\phi_{\mathbf{K}},\dot{\phi}_{\mathbf{K}}\}+\cdots\right)\right\rangle\right],\end{aligned} \tag{158}$$

where the second term is only non-zero if winding configurations are present. Continuing naturally,

$$\begin{aligned}\{f_0,\phi_{\mathbf{K}}\} &= -B\partial_{\mathbf{k}}f_0\times\partial_{\mathbf{k}}\phi_{\mathbf{K}} = B\delta(|\mathbf{k}|-k_F)\hat{n}\times\partial_{\mathbf{k}}\phi_{\mathbf{K}}\\ &= \frac{B}{k_F}\delta(|\mathbf{k}|-k_F)\partial_\theta\phi_{\mathbf{K}}\end{aligned} \tag{159}$$

where I used $f_0(\mathbf{k}) = \Theta(k_F-|\mathbf{k}|)$, and $\hat{n}\parallel\mathbf{k}$ so that $\hat{n}\times\partial_{\mathbf{k}} = \partial_\theta/k_F$ is the angular gradient at the FS set by the delta function. Note that I am leaving implicit a direction $\hat{z}$. This gives

$$\begin{aligned}S_{\mathrm{WZW}} &= -\sum_{\mathbf{K}\in\mathrm{mBZ}}\int dt\frac{d^2k}{2\pi B}\frac{B}{k_F}\delta(|\mathbf{k}|-k_F)\partial_\theta\phi_{\mathbf{K}}\left[\frac{1}{2}\dot{\phi}_{\mathbf{K}}+\frac{1}{6}\{\phi_{\mathbf{K}},\dot{\phi}_{\mathbf{K}}\}+\cdots\right]+S^{\mathrm{w}}_{\mathrm{WZW}}\\ &= -\sum_{\mathbf{K}\in\mathrm{mBZ}}\int dt\frac{d\theta}{2\pi}\partial_\theta\phi_{\mathbf{K}}(\theta)\left[\frac{1}{2}\dot{\phi}_{\mathbf{K}}(\theta)+\frac{1}{6}\{\phi_{\mathbf{K}}(\theta),\dot{\phi}_{\mathbf{K}}(\theta)\}+\cdots\right]+S^{\mathrm{w}}_{\mathrm{WZW}}\end{aligned} \tag{160}$$

where I defined for the time being

$$S^{\mathrm{w}}_{\mathrm{WZW}} \equiv -\sum_{\mathbf{K}\in\mathrm{mBZ}}\int dt\left[\left\langle f_0,\dot{\phi}_{\mathbf{K}}\right\rangle - \left\langle f_0, B(\partial_{\mathbf{k}}\times\partial_{\mathbf{k}}\phi_{\mathbf{K}})\left(\frac{1}{2}\dot{\phi}_{\mathbf{K}}+\frac{1}{6}\{\phi_{\mathbf{K}},\dot{\phi}_{\mathbf{K}}\}+\cdots\right)\right\rangle\right] \tag{161}$$

as the contribution to the winding term from S_{WZW}. As a final note, it is easy to see that

$$\begin{aligned}\{\phi_{\mathbf{K}}(\theta),\dot{\phi}_{\mathbf{K}}(\theta)\} = -B\partial_{\mathbf{k}}\phi_{\mathbf{K}}(\theta)\times\partial_{\mathbf{k}}\dot{\phi}_{\mathbf{K}}(\theta) &= -B\left[\partial_{|\mathbf{k}|}\hat{n}+\frac{1}{k_F}\partial_\theta\hat{\theta}\right]\phi_{\mathbf{K}}(\theta)\times\left[\partial_{|\mathbf{k}|}\hat{n}+\frac{1}{k_F}\partial_\theta\hat{\theta}\right]\dot{\phi}_{\mathbf{K}}(\theta)\\ &= 0,\end{aligned} \tag{162}$$

which also means all other higher-order terms vanish, giving in the semi-classical limit a very simple action

$$\begin{aligned}S_{\mathrm{WZW}} = &-\sum_{\mathbf{K}\in\mathrm{mBZ}}\int\frac{dtd\theta}{4\pi}\dot{\phi}_{\mathbf{K}}(\theta)\partial_\theta\phi_{\mathbf{K}}(\theta)\\ &-\sum_{\mathbf{K}\in\mathrm{mBZ}}\int dt\left[\left\langle f_0,\dot{\phi}_{\mathbf{K}}\right\rangle - \left\langle f_0, B(\partial_{\mathbf{k}}\times\partial_{\mathbf{k}}\phi_{\mathbf{K}})\left(\frac{1}{2}\dot{\phi}_{\mathbf{K}}+\frac{1}{6}\{\phi_{\mathbf{K}},\dot{\phi}_{\mathbf{K}}\}+\cdots\right)\right\rangle\right],\end{aligned} \tag{163}$$

The first term is off by a minus sign to Yuxuan's work [14], because they assume the θ angular variable is measured clockwise.

H-Term

The Hamiltonian part of the action (149) is

$$S_H = \sum_{\mathbf{K}\in\text{mBZ}} \int dt \left[\langle \{\phi_\mathbf{K}, f_0\}, \varepsilon(\mathbf{k})\rangle - \frac{1}{2}\langle \{\phi_\mathbf{K}, \{\phi_\mathbf{K}, f_0\}\}, \varepsilon(\mathbf{k})\rangle + \frac{1}{6}\langle \{\phi_\mathbf{K}, \{\phi_\mathbf{K}, \{\phi_\mathbf{K}, f_0\}\}\}, \varepsilon(\mathbf{k})\rangle + \cdots \right] \tag{164}$$

where I dropped the $\langle f_0, \varepsilon(\mathbf{k})\rangle$ term since it is of order ϕ^0 and is a shift to the energy. Note further from (159) that $\{\phi_\mathbf{K}, f_0\} \propto \delta(|\mathbf{k}| - k_F)\partial_\theta \phi_\mathbf{K}$, so the first term is a total derivative in θ (as $\varepsilon = \varepsilon(\mathbf{k})$). This further simplifies things to

$$S_H = -\sum_{\mathbf{K}\in\text{mBZ}} \int dt \left[-\frac{1}{2}\langle \{\{\phi_\mathbf{K}, f_0\}, \phi_\mathbf{K}\}, \varepsilon(\mathbf{k})\rangle + \frac{1}{6}\langle \{\{\phi_\mathbf{K}, \{\phi_\mathbf{K}, f_0\}\}, \phi_\mathbf{K}\}, \varepsilon(\mathbf{k})\rangle + \cdots \right], \tag{165}$$

where I reordered some brackets after dropping the total derivative. Flipping the Poisson bracket through (151), I find

$$\begin{aligned} S_H = &-\sum_{\mathbf{K}\in\text{mBZ}} \int dt \left[\frac{1}{2}\langle \{\phi_\mathbf{K}, f_0\}, \{\varepsilon(\mathbf{k}), \phi_\mathbf{K}\}\rangle - \frac{1}{6}\langle \{\phi_\mathbf{K}, \{\phi_\mathbf{K}, f_0\}\}, \{\varepsilon(\mathbf{k}), \phi_\mathbf{K}\}\rangle + \cdots \right] \\ &+ \sum_{\mathbf{K}\in\text{mBZ}} \int dt \left\langle B\partial_\mathbf{k} \times \partial_\mathbf{k}\phi_\mathbf{K} \left(\frac{1}{2}\{\phi_\mathbf{K}, f_0\} - \frac{1}{6}\{\phi_\mathbf{K}, \{\phi_\mathbf{K}, f_0\}\}\right), \varepsilon(\mathbf{k})\right\rangle, \end{aligned} \tag{166}$$

where the second term contains all of the winding contributions. The first term can be simplified by (159), giving

$$\begin{aligned} S_H = &\sum_{\mathbf{K}\in\text{mBZ}} \int dt \frac{d^2k}{2\pi B}\frac{B}{k_F}\delta(|\mathbf{k}| - k_F)\partial_\theta\phi_\mathbf{K} \left[\frac{1}{2}\{\varepsilon(\mathbf{k}), \phi_\mathbf{K}\} - \frac{1}{6}\{\phi_\mathbf{K}, \partial_\theta\phi_\mathbf{K}\}\{\varepsilon(\mathbf{k}), \phi_\mathbf{K}\} + \cdots \right] \\ &+ \sum_{\mathbf{K}\in\text{mBZ}} \int dt \left\langle B\partial_\mathbf{k} \times \partial_\mathbf{k}\phi_\mathbf{K} \left(\frac{1}{2}\{\phi_\mathbf{K}, f_0\} - \frac{1}{6}\{\phi_\mathbf{K}, \{\phi_\mathbf{K}, f_0\}\}\right), \varepsilon(\mathbf{k})\right\rangle \\ = &\sum_{\mathbf{K}\in\text{mBZ}} \int dt \frac{d\theta}{2\pi}\partial_\theta\phi_\mathbf{K}(\theta) \left[\frac{1}{2}\{\varepsilon(\mathbf{k}), \phi_\mathbf{K}(\theta)\} - \frac{1}{6}\{\phi_\mathbf{K}(\theta), \partial_\theta\phi_\mathbf{K}(\theta)\}\{\varepsilon(\mathbf{k}), \phi_\mathbf{K}(\theta)\} + \cdots \right] \\ &+ \sum_{\mathbf{K}\in\text{mBZ}} \int dt \left\langle B\partial_\mathbf{k} \times \partial_\mathbf{k}\phi_\mathbf{K} \left(\frac{1}{2}\{\phi_\mathbf{K}, f_0\} - \frac{1}{6}\{\phi_\mathbf{K}, \{\phi_\mathbf{K}, f_0\}\}\right), \varepsilon(\mathbf{k})\right\rangle, \end{aligned} \tag{167}$$

Note importantly that

$$\{\phi_\mathbf{K}(\theta), \partial_\theta\phi_\mathbf{K}(\theta)\} = 0, \tag{168}$$

for the same reason that (162) vanishes. Due to the nesting of this bracket (168) in all higher order terms not written explicitly, I find they also vanish. Fundamentally, this results in a non-interacting EFT in contrast to the interacting EFT without the magnetic field [3], and is a consequence of the independence of the cyclotron frequency on $|\mathbf{k}|$. For non-parabolic dispersion, this situation will likely change.

This leaves just leading term. Defining the cyclotron frequency as $\omega_c = B/m$, for a parabolic dispersion

$$\{\varepsilon(\mathbf{k}), \phi_\mathbf{K}(\theta)\} = -\frac{B}{k_F}\partial_\mathbf{k}\varepsilon(\mathbf{k})\Big|_{k_F} \partial_\theta\phi_\mathbf{K}(\theta)\hat{n} \times \hat{\theta} = -\omega_c\partial_\theta\phi_\mathbf{K}(\theta)\hat{z}, \tag{169}$$

and I conclude

$$\begin{aligned} S_H = &-\sum_{\mathbf{K}\in\text{mBZ}} \int dt \frac{d\theta}{4\pi}\omega_c \left(\partial_\theta\phi_\mathbf{K}(\theta)\right)^2 \\ &+ \sum_{\mathbf{K}\in\text{mBZ}} \int dt \left\langle B\partial_\mathbf{k} \times \partial_\mathbf{k}\phi_\mathbf{K} \left(\frac{1}{2}\{\phi_\mathbf{K}, f_0\} - \frac{1}{6}\{\phi_\mathbf{K}, \{\phi_\mathbf{K}, f_0\}\}\right), \varepsilon(\mathbf{k})\right\rangle, \end{aligned} \tag{170}$$

The second term is off by a minus sign to Yuxuan's work [14], but since this term drops out after imposing explicit winding configurations, the sign is irrelevant.

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