Solved Problems from Jackson's Classical Electrodynamics

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Last Update: Aug 2026

No form of AI was used in developing these solutions.

Chapter 1

Problem (1.3)

Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.

(a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

Solution: A spherical shell has the radial thickness of a point charge, implying that its charge distribution is related to that of a point charge, i.e. proportional to a delta function. By symmetry it's evident that this discontinuity isn't angular, but rather radial. So, take the ansatz $\rho(r) = C\delta(r - R)$ and solve for C such that the total charge is Q,

$$Q = \int dV \; \rho = C \int r^2 dr d\Omega \; \delta(r - R) = C \left(4\pi R^2\right) \implies C = \frac{Q}{4\pi R^2}, \tag{1.3-1}$$

$$\therefore \; \boxed{\rho(r) = \frac{Q}{4\pi R^2}\delta(r - R)} \tag{1.3-2}$$

(b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

Solution: This problem is similar to (a). By cylindrical symmetry and that fact that it is a surface, $\rho(s) = C\delta(s - b)$, where s is the radius in cylindrical coordinates. In the same way,

$$Q = C \int s ds d\theta dz \; \delta(s - b) = C \left(2\pi b\right) \int dz. \tag{1.3-3}$$

We are given charge per unit length $\lambda = Q/L$, so the seemingly divergent integral (1.3-3) is irrelevant and I find

$$\lambda = C(2\pi b) \implies C = \frac{\lambda}{2\pi b}, \; \therefore \; \boxed{\rho(s) = \frac{\lambda}{2\pi b}\delta(s - b)} \tag{1.3-4}$$

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disk of negligible thickness and radius R.

Solution: We are no longer working with surfaces/shells, and we have a more continuous distribution. The charge density must be proportional to a Heaviside function that is 1 inside the disk and 0 outside. In addition, the thickness is negligible and must be accounted for via $\delta(z)$. Thus, $\rho(z, s) = C\delta(z)\theta(R - s)$ such that

$$\begin{aligned} Q = C \int s ds d\theta dz \; \delta(z)\theta(R - s) &= C \left(2\pi\right) \int dz \; \delta(z) \int_0^R s ds \\ &= C \left(\pi R^2\right) \implies C = \frac{Q}{\pi R^2}, \end{aligned} \tag{1.3-5}$$

$$\therefore \; \boxed{\rho(z, s) = \frac{Q}{\pi R^2}\delta(z)\theta(R - s)} \tag{1.3-6}$$

(d) The same as part (c), but using spherical coordinates.

Solution: This follows directly from (1.3-6) with the replacement $s \to r$ and $z = r\cos\theta$, where

$$\delta(z) = \delta(r\cos\theta) = \frac{\delta(\theta - \pi/2)}{|-r\sin(\pi/2)|} = \frac{\delta(\theta-\pi/2)}{r}, \tag{1.3-7}$$

such that

$$\boxed{\rho(r,\theta) = \frac{Q}{\pi R^2}\frac{\theta(R-r)\delta(\theta-\pi/2)}{r}} \tag{1.3-8}$$

where I used the formula for $\delta(f(x))$.

Jackson (1.5)

The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right), \tag{1.5-1}$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

Solution: This problem is simple in the sense that the solution method is obvious; use rewrite Poisson's equation for the charge density, $\rho = -\epsilon_0\nabla^2\Phi$. However, one must be subtle with the computation of the Laplacian with the $1/r$ factor being present, as $\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r})$. So, my method is to use the Laplacian in spherical coordinates (as a function of radius) and carefully isolate the $1/r$ term:

$$\begin{aligned}
\rho = -\frac{q}{4\pi}\nabla^2\left(\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)\right) &= -\frac{q}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)\right)\right] \\
&= -\frac{q}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left[r\frac{d}{dr}\left(e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right) + r^2 e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\frac{d}{dr}\left(\frac{1}{r}\right)\right] \\
&= -\frac{q}{4\pi}e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right] \\
&\quad - \frac{q}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left[r\frac{d}{dr}\left(e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right)\right] \\
&\quad - \frac{q}{4\pi}\frac{d}{dr}\left(\frac{1}{r}\right)\frac{d}{dr}\left(e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right).
\end{aligned} \tag{1.5-2}$$

Success. Since I know that

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right] = \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{r}), \tag{1.5-3}$$

(1.5-2) can be rewritten as

$$\rho = q\delta^3(\mathbf{r}) - \frac{q}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left[r\frac{d}{dr}\left(e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right)\right] - \frac{q}{4\pi}\frac{d}{dr}\left(\frac{1}{r}\right)\frac{d}{dr}\left(e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right), \tag{1.5-4}$$

where I used the fact that the delta function sets $r = 0$ and the pre-factor goes to 1. Continuing with the expansion,

$$\begin{aligned}
\rho &= q\delta^3(\mathbf{r}) - \frac{q}{4\pi}\frac{1}{r}\frac{d^2}{dr^2}\left(e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right) - \cancel{\frac{q}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left(e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right)} \\
&\quad + \cancel{\frac{q}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left(e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right)} \\
&= q\delta^3(\mathbf{r}) - \frac{q}{4\pi}\frac{1}{r}\frac{d^2}{dr^2}\left(e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right) \\
&= q\delta^3(\mathbf{r}) - \frac{q}{4\pi}\frac{1}{r}\frac{d}{dr}\left(-\alpha e^{-\alpha r}\left(\frac{1}{2}+\frac{\alpha r}{2}\right)\right) \\
&= q\delta^3(\mathbf{r}) - \frac{q}{4\pi}\frac{1}{r}\left(\alpha^2 e^{-\alpha r}\left(\frac{1}{2}+\frac{\alpha r}{2}\right) - \frac{\alpha^2}{2}e^{-\alpha r}\right).
\end{aligned} \tag{1.5-5}$$

To terms cancel inside the parentheses, leaving just

$$\boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r}) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}} \tag{1.5-6}$$

as the charge density giving rise to a time-averaged (neutral) hydrogen atom potential (1.5-1). Which has discrete (delta function) and continuous contributions.

The interpretation of this charge density is easy if you know what a neutral hydrogen atom is; $q\delta^3(\mathbf{r})$ is the charge density of a point charge at the origin with charge $+q$, and is representative of the proton. The other term with charge $-q$ is the "cloud" of electrons, which decays as $r \to \infty$, semi-representative of the fact that the probability density of electrons in quantum mechanics is larger near the origin. Is the net charge really zero?

$$Q = \int dV\ \rho(\mathbf{r}) = q\int dV\ \delta^3(\mathbf{r}) - \frac{q\alpha^3}{8\pi}(4\pi)\int_0^\infty dr\ r^2 e^{-\alpha r} = q - \frac{q\alpha^3}{2}\frac{1}{\alpha^3}\Gamma(3) = 0, \tag{1.5-7}$$

It is! Note that $\Gamma(3) = 2$.

Jackson (1.10)

Prove the mean value theorem: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.

Solution: I could re-derive the expansion of the electrostatic potential in terms of Green's functions, but for this problem I can just take it to be true. As is usual, let $\phi = \Phi$ and $\psi = G$ and substitute them into Green's theorem Eq. (1.35) of the text,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\ \rho(\mathbf{x}')G(\mathbf{x};\mathbf{x}') + \frac{1}{4\pi}\oint_{\partial V} d^2x'\ \left[G(\mathbf{x};\mathbf{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x};\mathbf{x}')}{\partial n'}\right], \tag{1.10-1}$$

which is equivalent to (1.42) of the text, and is a consequence of the Green's function choice

$$G(\mathbf{x};\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x};\mathbf{x}'). \qquad (\nabla^2 F = 0) \tag{1.10-2}$$

I am now in a position to discuss the problem at hand. For charge-free space note that $\rho(\mathbf{x}') = 0$ which results in

$$\Phi(\mathbf{x}) = \frac{1}{4\pi}\oint_{\partial V} d^2x'\ \left[G(\mathbf{x};\mathbf{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x};\mathbf{x}')}{\partial n'}\right]. \tag{1.10-3}$$

Recall from Griffith's that $\sigma = -\epsilon_0\frac{\partial\Phi}{\partial n'}$ is the induced surface charge density on the surface $\vec{n}$, which must be zero since we are in charge-free space, leaving only a single term:

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_{\partial V} d^2x'\ \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x};\mathbf{x}')}{\partial n'}. \tag{1.10-4}$$

This discussion is equivalent to stating that the Green's function obeys Dirichlet boundary conditions. Since the Green's function depends only on $\mathbf{x}-\mathbf{x}'$ in flat spacetime, I can choose the origin to be the point at which the potential is measured, $\mathbf{x} = 0$, and the result will be valid $\forall\mathbf{x}$. With this choice, I find that

$$G(0,\mathbf{x}') = \frac{1}{x'} + F(\mathbf{x},\mathbf{x}'). \tag{1.10-5}$$

To satisfy the aforementioned Dirichlet boundary conditions, the Green's function must vanish at some spherical surface of radius R, implying that $F = -1/R$ is a constant function and therefore irrelevant to the physics. Why is it irrelevant? Well, I need to compute the normal derivative of G,

$$\frac{\partial G(\mathbf{x};\mathbf{x}')}{\partial n'} = \vec{\nabla}'\left(\frac{1}{x'} - \frac{1}{R}\right)\cdot\hat{n}\Big|_{x'=R} = -\frac{1}{x'^2}(\hat{x}'\cdot\hat{n})\Big|_{x'=R} = -\frac{1}{R^2}, \tag{1.10-6}$$

which is a constant over the surface. Thus, (1.10-4) becomes

$$\boxed{\Phi(x) = \frac{1}{4\pi R^2}\oint_{\partial V} d^2x'\ \Phi(\mathbf{x}') \qquad \square} \tag{1.10-7}$$

which states that the potential at some point $\mathbf{x}$, $\Phi(\mathbf{x})$, is the average of the potential over any spherical surface ∂V with radius R centered about that point.

Jackson (1.12)

Prove Green's reciprocation theorem: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V d^3r\ \rho\Phi' + \oint_{\partial V} da\ \sigma\Phi' = \int_V d^3r\ \rho'\Phi + \oint_{\partial V} da\ \sigma'\Phi. \tag{1.12-1}$$

Solution: This is an immediate consequence of Green's 2nd identity with the choice $A = \Phi(\mathbf{r})$ and $\mathbf{B} = \Phi'(\mathbf{r})$,

$$\int_V d^3r \left[\Phi(\mathbf{r})\left(-\frac{\rho'(\mathbf{r})}{\epsilon_0}\right) - \Phi'(\mathbf{r})\left(-\frac{\rho(\mathbf{r})}{\epsilon_0}\right)\right] = \oint_{\partial V} \mathbf{da}\cdot\left[\Phi(\mathbf{r})\nabla\Phi'(\mathbf{r}) - \Phi'(\mathbf{r})\nabla\Phi(\mathbf{r})\right], \tag{1.12-2}$$

where since on the surface ∂V it is valid that $\hat{n}\cdot\nabla\Phi = -\frac{\sigma}{\epsilon_0}$, a cancellation of constant factors and rearranging gives (1.12-1).

Jackson (1.13)

Two infinite grounded parallel conducting planes are separated by a distance d. A point charge q is placed between the planes. Use the reciprocation theorem of Green to prove that the total induced charge on one of the planes is equal to $(-q)$ times the fractional perpendicular distance of the point charge from the other plane. (Hint: As your comparison electrostatic problem with the same surfaces choose one whose charge densities and potential are known and simple.)

Solution: The reciprocation theorem of Green is given by

$$\int_V d^3x\ \rho\Phi' + \int_{\partial V} d^2x\ \sigma\Phi' = \int_V d^3x\ \rho'\Phi + \int_{\partial V} d^2x\ \sigma'\Phi. \tag{1.13-1}$$

This theorem is valid for any two pairs of charge densities and corresponding potentials ρ, σ, Φ, and ρ', σ', Φ' iff they are contained in the same volume V and boundary surface ∂V. So, my idea is to construct a second set of charge densities ρ', σ' with corresponding potentials Φ' that are simple, so that I may solve for the induced charge density σ of the given problem.

Inspect the figure below. On the left is the system of the given problem, with a point charge q a distance $\widetilde{x}$ from sheet (1). A simpler problem that has the same volume and boundary is given on the right, which is the case of two (infinite) parallel-plate capacitors, with plate (1) at $x = 0$ and having surface charge density $+\sigma'$, and plate (2) at $x = d$ and having surface charge density $-\sigma'$. Gauss' Law tells me that

$$\mathbf{E}' = \frac{\sigma'}{\epsilon_0}\hat{x} \implies \Phi' = -\frac{\sigma'}{\epsilon_0}\int_0^x dx' = -\frac{\sigma'}{\epsilon_0}x \tag{1.13-2}$$

is the electric field between the plates, and zero outside the plates (no net charge). So, I have solved the system I just constructed: $\rho' = 0$, $\sigma' = \sigma'$ and $\Phi' = -\sigma' x/\epsilon_0$. These values allow me to write (1.13-1) as

$$\begin{aligned} -\frac{\sigma'}{\epsilon_0}\int_V d^3x\ \rho x + \int_{\partial V} d^2x\ \sigma\Phi' &= \sigma'\int_{\partial V} d^2x\ \Phi \\ &= 0, \end{aligned} \tag{1.13-3}$$

where in the last step I made the use of the fact that in the original problem $\Phi = 0$ on the surface ∂V since they are conductors. I am measuring the potential Φ' relative to plate (1), so the surface integral only has the contribution of plate

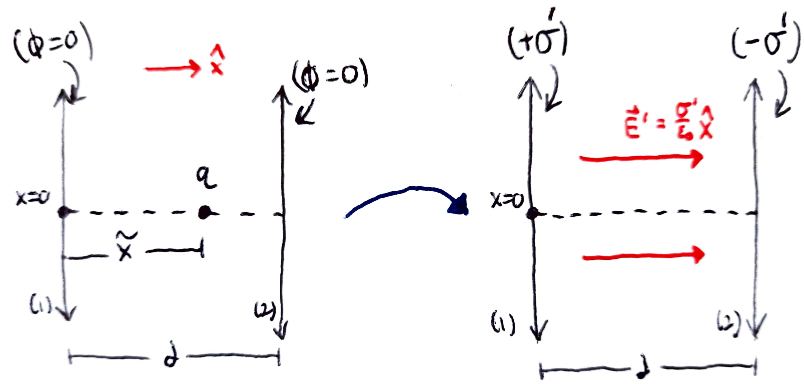


Figure 1: Jackson 1.13

(2). In addition, the original problem only has the charge density $\rho(x,y,z) = q\delta(x-\widetilde{x})\delta(y)\delta(z)$, such that (1.13-3) can be written as

$$\begin{aligned}
0 &= -\frac{q\sigma'}{\epsilon_0}\int_V d^3x\ x\delta(x-\widetilde{x})\delta(y)\delta(z) + \int_{(2)} d^2x\ \sigma_{(2)}\left(-\frac{\sigma'}{\epsilon_0}d\right) \\
&= -\frac{q\sigma'}{\epsilon_0}\widetilde{x} - \frac{\sigma'}{\epsilon_0}d\int_{(2)} d^2x\ \sigma_{(2)} \\
&= -\frac{\sigma'}{\epsilon_0}\left(q\widetilde{x} + q_{\text{ind.}}d\right) \implies \boxed{q_{\text{ind.}} = -q\frac{\widetilde{x}}{d}}
\end{aligned} \tag{1.13-4}$$

where I interpretted the surface integral of the charge density as the total charge induced on plate (2), and noted that σ' is arbitrary and can be chosen to be non-zero.

Therefore, the charge induced on plate (2) is $-q$ times the fractional distance $\widetilde{x}/d$ of the point charge to plate (1). Since the original problem is reflection symmetric about a line between (and parallel) to the conducting sheets, this argument is valid for either conducting sheet, and hence the problem is proved.

Jackson (1.15)

Prove Thompson's theorem: If a number of surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors.

Solution: Take (1.54) of Jackson and perform a variation:

$$\begin{aligned}
W = \frac{\epsilon_0}{2}\int_V d^3r|\nabla\Phi|^2 \implies \delta W &= \epsilon_0\int_V d^3r\ \nabla\Phi\cdot\nabla\delta\Phi \\
&= \epsilon_0\oint_{\partial V} d^2x\ \hat{n}\cdot\nabla\Phi\delta\Phi - \epsilon_0\int_V d^3r\ \nabla^2\Phi\delta\Phi.
\end{aligned} \tag{1.15-1}$$

Assuming the charge lives only on the surface, then $\nabla^2\Phi \to 0$ and we are left with

$$\delta W = \epsilon_0\oint_{\partial V} d^2x\ \hat{n}\cdot\nabla\Phi\delta\Phi. \tag{1.15-2}$$

If the energy in the region is at an absolute minimum, $\delta W = 0$. This should be valid $\forall\delta\Phi$, implying that the normal derivative (or gradient) of the potential on ALL of the surfaces must vanish. This is equivalent to the statement that every surface is an equipotential, and hence Thompson's theorem is proved. $\square$

Jackson (1.17)

A volume V in vacuum is bounded by a surface S consisting of several separate conducting surfaces S_i. One conductor is held at unit potential and all the other conductors at zero potential.

(a) Show that the capacitance of the one conductor is given by

$$C = \epsilon_0 \int_V d^3x |\nabla\Phi|^2,$$

where $\Phi(\mathbf{x})$ is the solution for the potential.

Solution: To solve this problem I will use Green's 1st identity (or I.B.P),

$$\int_V d^3x \left[A\nabla^2 B + \nabla A \cdot \nabla B\right] = \oint_{\partial V} (\mathbf{da} \cdot \nabla B)\, A, \tag{1.17-1}$$

which if I note $|\nabla\Phi|^2 = \nabla\Phi \cdot \nabla\Phi$ allows me to express the following integral as

$$\epsilon_0 \int_V d^3x |\nabla\Phi|^2 = \epsilon_0 \oint_{\partial V} (\mathbf{da} \cdot \nabla\Phi)\, \Phi - \int_V d^3x\ \Phi\nabla^2\Phi. \tag{1.17-2}$$

Note importantly that

$$\mathbf{da} \cdot \nabla\Phi = da\frac{\partial\Phi}{\partial n} = da\left(\frac{\sigma}{\epsilon_0}\right), \tag{1.17-3}$$

where I recognized the normal derivative of Φ as a variant on the induced charge density. In addition, since the volume is in vacuum, $\nabla^2\Phi = -\rho/\epsilon_0 = 0$, so the volume integral in (1.17-2) vanishes:

$$\epsilon_0 \int_V d^3x |\nabla\Phi|^2 = \oint_{\partial V} da\ \Phi\sigma. \tag{1.17-4}$$

The resulting integral is over the entire boundary, and is greatly simplified by noting that $\Phi = 1$ on our special surface and $\Phi = 0$ on all other surfaces. Thus, defining σ as the surface charge on the conductor at unit potential, I find

$$\epsilon_0 \int_V d^3x |\nabla\Phi|^2 = \oint_{\partial V} da\ (1)\sigma = Q \tag{1.17-5}$$

where Q is the total charge present on the unit potential conductor. Since the capacitance is defined as "the total charge present on a conductor held at unit potential", I can recognize the r.h.s as the capacitance,

$$\boxed{\epsilon_0 \int_V d^3x |\nabla\Phi|^2 = C} \tag{1.17-6}$$

and hence the relation given is proved. □

(b) Show that the true capacitance C is always less than or equal to the quantity

$$C[\Psi] = \epsilon_0 \int_V d^3x |\nabla\Psi|^2,$$

where Ψ is any trial function satisfying the boundary conditions on the conductors. This is a variational principle for the capacitance that yields an upper bound.

Solution: If this is to be compared to the system in part (a), showing that this is an upper-bound to that capacitance, it makes sense to take Ψ as a variation around the potential of part (a): $\Psi = \Phi + \delta\Psi$. This is further justified since Ψ must satisfy the same boundary conditions as Φ, further implying that $\delta\Psi = 0$ on the boundary. We have then, that

$$\begin{aligned} |\nabla\Psi|^2 = \nabla\Psi \cdot \nabla\Psi &= (\nabla\Phi + \nabla\delta\Psi) \cdot (\nabla\Phi + \nabla\delta\Psi) \\ &= |\nabla\Phi|^2 + 2\nabla\Phi \cdot \nabla\delta\Psi + |\nabla\delta\Psi|^2, \end{aligned} \tag{1.17-7}$$

such that

$$
\begin{aligned}
C[\Psi] &= \epsilon_0 \int_V d^3x |\nabla\Phi|^2 + 2\epsilon_0 \int_V d^3x \nabla\Phi \cdot \nabla\delta\Psi + \epsilon_0 \int_V d^3x |\nabla\delta\Psi|^2 \\
&= C + 2\epsilon_0 \int_V d^3x \nabla\Phi \cdot \nabla\delta\Psi + \epsilon_0 \int_V d^3x |\nabla\delta\Psi|^2
\end{aligned}
\tag{1.17-8}
$$

where I recognized $C[\Phi] = C$ as the true capacitance. I will now expand the middle integral using Green's 1st Identity (or I.B.P):

$$
C[\Psi] = C + 2\epsilon_0 \left(\oint_{\partial V} (\mathbf{da} \cdot \nabla\Phi)\, \delta\Psi - \int_V d^3x \nabla^2\Phi\delta\Psi \right) + \epsilon_0 \int_V d^3x |\nabla\delta\Psi|^2 \tag{1.17-9}
$$

Since I argued that $\delta\Psi = 0$ on the boundary, the surface integral vanishes. Furthermore, $\nabla^2\Phi = 0$ in vacuum, so only one integral remains:

$$
C[\Psi] = C + \epsilon_0 \int_V d^3x |\nabla\delta\Psi|^2 \tag{1.17-10}
$$

For any arbitrary (nonzero) shift, whether positive or negative, the remaining integral is strictly positive. It is therefore valid for me to claim that

$$
\boxed{C \leq C[\Psi]} \tag{1.17-11}
$$

and hence $C[\Psi]$ is an upper-bound on the true capacitance. □

Chapter 2

Jackson (2.2)

Using the method of images, discuss the problem of a point charge q inside a hollow, grounded, conducting sphere of inner radius a. Find

(a) The potential inside the sphere;

Solution: Inspect the following Figure:

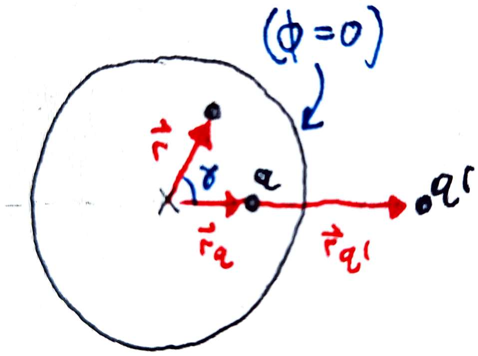


Figure 2: Jackson (2.2)

Here, I let $\mathbf{r}$ be the position (inside the sphere) at which the potential is evaluated, $\mathbf{r}_q$ is position of the charge q, and $\mathbf{r}_{q'}$ is the position of the image charge q'. My choice in choosing $\mathbf{r}_{q'} \parallel \mathbf{r}_q$ is through the symmetry of the system. The potential due to these two charges has the simple form

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{||\mathbf{r}-\mathbf{r}_q||} + \frac{q'}{||\mathbf{r}-\mathbf{r}_{q'}||}\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 - 2rr_q\cos\gamma + r_q^2}} + \frac{q'}{\sqrt{r^2 - 2rr_{q'}\cos\gamma + r_{q'}^2}}\right],\end{aligned} \tag{2.2-1}$$

where I am defining $r = |\mathbf{r}|$ as the magnitude of a vector, and taking $\cos\gamma = \hat{r}\cdot\hat{r}_q$ as the angle which subtends $\mathbf{r}$ and $\mathbf{r}_q$. For a sphere radius of a, the condition that the potential lies inside implies the restriction $r < a$. It must be true that $\Phi(a) = 0\ \forall\gamma$, and since I have two undetermined quantities, I will choose the simple cases of $\gamma = 0, \pi$:

$$\begin{aligned}(\gamma = 0), \qquad 0 &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(a-r_q)^2}} + \frac{q'}{\sqrt{(a-r_{q'})^2}}\right] = q(r_{q'} - a) + q'(a - r_q), \\ (\gamma = \pi), \qquad 0 &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(a+r_q)^2}} + \frac{q'}{\sqrt{(a+r_{q'})^2}}\right] = q(r_{q'} + a) + q'(a + r_q),\end{aligned} \tag{2.2-2}$$

where I made use of the fact that $r_q < a < r_{q'}$. If these two equations are added I find

$$0 = 2qr_{q'} + 2q'a \implies q' = -\frac{qr_{q'}}{a}, \tag{2.2-3}$$

which can be substituted into the $\gamma = 0$ equation to give

$$0 = q(r_{q'} - a) - \frac{qr_{q'}}{a}(a - r_{q'}) \implies r_{q'} = \frac{a^2}{r_q}, \tag{2.2-4}$$

which uniquely determines the position of the image charge. This representation of $r_{q'}$ implies that the magnitude of the image charge is $q' = -qa/r_q$. With this choice, I will determine if $\Phi(a) = 0\ \forall\gamma$:

$$\begin{aligned}
\Phi(a) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2 - 2ar_q\cos\gamma + r_q^2}} - \frac{qa}{r_q\sqrt{a^2 - 2a\left(\frac{a^2}{r_q}\right)\cos\gamma + \left(\frac{a^2}{r_q}\right)^2}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2 - 2ar_q\cos\gamma + r_q^2}} - \frac{q}{\sqrt{a^2 - 2ar_q\cos\gamma + r_q^2}}\right] \\
&= 0,
\end{aligned} \tag{2.2-5}$$

which is elegant. Therefore,

$$\boxed{\Phi(r,\gamma) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 - 2rr_q\cos\gamma + r_q^2}} - \frac{a/r_q}{\sqrt{r^2 - 2r\left(\frac{a^2}{r_q}\right)\cos\gamma + \left(\frac{a^2}{r_q}\right)^2}}\right]} \tag{2.2-6}$$

is the potential in spherical coordinates inside the sphere.

(b) The induced surface-charge density;

Solution: Recall that

$$\sigma(\mathbf{r}) = \epsilon_0\hat{r}\cdot\nabla\Phi(r,\gamma)\Big|_{r=a} \tag{2.2-7}$$

where I used $\hat{n} = -\hat{r}$ for the inside surface. Note that I took $\mathbf{E}_{\text{out}} = 0$ since its a conductor and the charge is inside. So, since the surface-charge density only picks out the radial component, I only need to evaluate the radial component of the gradient in spherical coordinates, $\nabla \sim \hat{r}\partial_r$. It is standard to compute the radial derivative then evaluate it on the surface,

$$\begin{aligned}
\partial_r\Phi(r,\gamma)\Big|_{r=a} &= -\frac{q}{4\pi\epsilon_0}\left[\frac{r - r_q\cos\gamma}{\left(r^2 - 2rr_q\cos\gamma + r_q^2\right)^{3/2}} - \left(\frac{a}{r_q}\right)\frac{r - \frac{a^2}{r_q}\cos\gamma}{\left(r^2 - 2r\left(\frac{a^2}{r_q}\right)\cos\gamma + \left(\frac{a^2}{r_q}\right)^2\right)^{3/2}}\right]\Bigg|_{r=a} \\
&= -\frac{q}{4\pi\epsilon_0}\left[\frac{a - r_q\cos\gamma}{\left(a^2 - 2ar_q\cos\gamma + r_q^2\right)^{3/2}} - \left(\frac{a}{r_q}\right)\frac{a - \frac{a^2}{r_q}\cos\gamma}{\left(a^2 - 2a\left(\frac{a^2}{r_q}\right)\cos\gamma + \left(\frac{a^2}{r_q}\right)^2\right)^{3/2}}\right] \\
&= -\frac{q}{4\pi\epsilon_0}\left[\frac{a - \frac{r_q^2}{a}}{\left(a^2 - 2ar_q\cos\gamma + r_q^2\right)^{3/2}}\right],
\end{aligned} \tag{2.2-8}$$

where I was able to combine the two fractions via a similar manner as (2.2-5). In light of (2.2-7), the surface charge density has the form

$$\boxed{\sigma(\gamma) = -\frac{qa}{4\pi}\left[\frac{1 - \frac{r_q^2}{a^2}}{\left(a^2 - 2ar_q\cos\gamma + r_q^2\right)^{3/2}}\right]}. \tag{2.2-9}$$

which is indeed negative since $r_q < a$. Now, since I now the electric field is outside, the total induced charge must be negative that of the enclosed point charge, $-q$. Is that true?

$$
\begin{aligned}
Q = \int da\sigma(\gamma) = -\frac{q}{4\pi}(2\pi)\left(a - \frac{r_q^2}{a}\right)\int_{-1}^{1}\frac{d(\cos\gamma)}{\left(a^2 - 2ar_q\cos\gamma + r_q^2\right)^{3/2}} &= -\frac{q}{2}\left(\frac{a - \frac{r_q^2}{a}}{2ar_q}\right)(-2)\left(\frac{1}{a + r_q} - \frac{1}{a - r_q}\right) \\
&= \frac{q}{2r_q}\left(a^2 - r_q^2\right)\left(\frac{-2r_q}{a^2 - r_q^2}\right) \\
&= -q, \quad \square
\end{aligned}
\tag{2.2-10}
$$

of course it is.

(c) The magnitude and direction of the force acting on q.

Solution: I could evaluate this by writing the differential force $d\mathbf{F}$ in terms of the surface-charge density (2.2-9) and doing the integral, but instead I know that the system is equivalent to the given charge q and its image q', so I just to use Coulomb's Law for the force due to the image $q' = -qa/r_q$ at a distance of $\frac{a^2}{r_q} - r_q$ from the charge q:

$$
\mathbf{F} = \frac{q^2}{4\pi\epsilon_0}\left(\frac{a}{r_q\left(\frac{a^2}{r_q} - r_q\right)^2}\right)\hat{r}_q \implies \boxed{\mathbf{F} = \frac{q^2}{4\pi\epsilon_0}\left(\frac{r_q}{a^3}\right)\frac{1}{\left(1 - \left(r_q/a\right)^2\right)^2}\hat{r}_q}
\tag{2.2-11}
$$

which is an attractive force pushing the charge towards the point of the sphere's inner surface with the maximal charge density $\sigma(0)$.

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?

Solution: If the sphere is kept a fixed potential V, I need to add a term to the potential I derived (2.6). Since this addition is constant over the whole surface, the surface-charge density is left invariant, as it is proportional to a derivative of the potential. Hence, the solution is unchanged.

In the case that a total charge of Q is on the surface, I know that we must still have $-q$ as the inner surface charge for Gauss' Law to be satisfied. This fixes the outer surface charge as $Q + q$, which is a constant, giving rise to a (constant) addition to the potential which (using the same argument above) leaves the physics unchanged.

Jackson (2.5)

This question is in two parts; You might find it useful to note the identity

$$\frac{1}{z(z^2-1)^2} + \frac{1}{z^3(z^2-1)} = -\frac{1}{z^3} + \frac{1}{4(z-1)^2} - \frac{1}{4(z+1)^2}.$$

(a) Show that the work done to remove the charge q from a distance $r > a$ to infinity against the force, (Jackson 2.6), of a grounded conducting sphere is

$$W = \frac{q^2 a}{8\pi\epsilon_0(r^2 - a^2)}.$$

Relate this result to the electrostatic potential, (Jackson 2.3), and the energy discussion of Section 1.11.

Solution: The force (Jackson 2.6) is

$$|\mathbf{F}| = \frac{q^2 a}{4\pi\epsilon_0}\frac{1}{y^3}\left(1 - \frac{a^2}{y^2}\right)^{-2}. \tag{2.5-1}$$

The work done as I move the charge under this force is given by the usual formula:

$$\begin{aligned} W = \int dy|\mathbf{F}| &= \frac{q^2 a}{4\pi\epsilon_0}\int_r^\infty \frac{dy}{y^3\left(1-\frac{a^2}{y}\right)^2} \\ &= \frac{q^2 a}{4\pi\epsilon_0}\int_r^\infty \frac{dy\ y}{(y^2-a^2)^2} \\ &= \frac{q^2 a}{8\pi\epsilon_0}\int_{r^2-a^2}^\infty \frac{du}{u^2} \\ &= \frac{q^2 a}{8\pi\epsilon_0(r^2-a^2)}\ \Box \end{aligned} \tag{2.5-2}$$

Using the values $q' = qa/y$ and $y' = a^2/y$, the potential (Jackson 2.6) can be written as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{||\mathbf{x}-\mathbf{y}||} + \frac{q'}{||\mathbf{x}-\mathbf{y}'||}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{||\mathbf{x}-\mathbf{y}||} - \frac{a}{y||\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}||}\right). \tag{2.5-3}$$

I will now make the choice $\mathbf{x} = \mathbf{y} = \mathbf{r}$, as the charge corresponding to (2.5-3) is the one we moving. The first term appears to cause issues, but can be ignored since it is an infinite self-energy:

$$\Phi(\mathbf{r}) = -\frac{q}{4\pi\epsilon_0}\frac{a}{y||\mathbf{r}-\frac{a^2}{y^2}\mathbf{r}||} = -\frac{qa}{4\pi\epsilon_0(r^2-a^2)}. \tag{2.5-4}$$

Recognizing $W = q\Phi$, I get the form (2.5-2) up to a factor of $-1/2$. In reality, I am double counting in (2.5-5) by considering the effects of the image charge, and I should divide the resulting work by 1/2, which gives the work formula (2.5-2). This is because it takes no work to move an image charge. The minus sign difference is because in one case I am removing the charge, while in the other I am placing it in the system, which is redundant for my purposes.

(b) Repeat the calculation of the work done to remove the charge q against the force, (Jackson 2.9), of an isolated charged conducting sphere. Show that the work done is

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2(r^2-a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right].$$

Relate the work to the electrostatic potential, (Jackson 2.8), and the energy discussion of Section 1.11.

Solution: I perform the exact same calculation as before, but with the new force

$$|\mathbf{F}| = \frac{q}{4\pi\epsilon_0}\frac{1}{y^2}\left[Q - \frac{qa^3(2y^2-a^2)}{y(y^2-a^2)^2}\right], \tag{2.5-5}$$

from which follows that

$$
\begin{aligned}
W = -\int dy|\mathbf{F}| &= -\frac{q}{4\pi\epsilon_0}\int_r^\infty dy\ \frac{1}{y^2}\left[Q - \frac{qa^3(2y^2-a^2)}{y(y^2-a^2)^2}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left(-\frac{qQ}{r}\right) + \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{a}\right)\int_r^\infty dy \frac{2y^2-a^2}{y^3(y^2/a^2-1)^2}.
\end{aligned}
\tag{2.5-6}
$$

Let $z = y/a$, so $dy = adz$ and

$$
W = \frac{1}{4\pi\epsilon_0}\left(-\frac{qQ}{r}\right) + \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{a}\right)\int_{r/a}^\infty dz \frac{2z^2-1}{z^3(z^2-1)^2}.
\tag{2.5-7}
$$

Using the partial fraction decomposition given in the problem statement,

$$
\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\left(-\frac{qQ}{r}\right) + \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{a}\right)\int_{r/a}^\infty dz\ -\frac{1}{z^3} + \frac{1}{4(z-1)^2} - \frac{1}{4(z+1)^2} \\
&= \frac{1}{4\pi\epsilon_0}\left(-\frac{qQ}{r}\right) + \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{a}\right)\left[\frac{1}{2z^2} - \frac{1}{4(z-1)} + \frac{1}{4(z+1)}\right]\Big|_{r/a}^\infty \\
&= \frac{1}{4\pi\epsilon_0}\left(-\frac{qQ}{r}\right) - \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{a}\right)\left[\frac{a^2}{2r^2} - \frac{a}{4(r-a)} + \frac{a}{4(r+a)}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q^2a}{2(r^2-a^2)} - \frac{q^2a}{2r^2} - \frac{qQ}{r}\right] \\
&= -\frac{1}{4\pi\epsilon_0}\left[\frac{q^2a^3}{2r^2(r^2-a^2)} + \frac{qQ}{r}\right] \quad \square
\end{aligned}
\tag{2.5-8}
$$

which is precisely the same as the problem statement. The potential (Jackson 2.8) has the form

$$
\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{||\mathbf{x}-\mathbf{y}||} - \frac{qa}{y||\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}||} + \frac{Q+qa/y}{||\mathbf{x}||}\right],
\tag{2.5-9}
$$

which for our purposes can be simplified analogously as (2.5-4) be letting $\mathbf{x} = \mathbf{y} = \mathbf{r}$ and dropping the infinite self-energy term. Recall that the second term gives rise to the same (2.5-4), such that

$$
\Phi(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0}\left[\frac{qa}{(r^2-a^2)} - \frac{qa}{r^2} - \frac{Q}{r}\right].
\tag{2.5-10}
$$

Recognizing that $W = q\Phi$, (2.5-10) gives rise to the same work if we account for the double counting on the terms that are a result of the image charge (the ones with q^2). The minus sign difference is because in one case I am removing the charge and in the other I am putting it in, and is redundant.

Jackson (2.7)

Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

(a) Write down the appropriate Green function $G(\mathbf{x}; \mathbf{x}')$.

Solution: To satisfy the given boundary conditions, $G(\mathbf{x}; \mathbf{x}') = 0$ on the plane and at infinity. The most general choice,

$$G(\mathbf{x}; \mathbf{x}') = \frac{1}{||\mathbf{x} - \mathbf{x}'||} + F(\mathbf{x}; \mathbf{x}'), \tag{2.7-1}$$

can be determined to vanish at the boundary through an argument akin to the method of images. This is because $G(\mathbf{x}; \mathbf{x}')$ is proportional to the $\Phi(\mathbf{x})$ via a point charge at $\mathbf{x}'$. If I envision a point charge a distance $+z'$ from a conducting plane, the method of images implies the placement of an image with opposite sign a distance $-z'$ from the plane. So, first using the formula for $||\mathbf{x} - \mathbf{x}'||$ in cylindrical coordinates,

$$G(\mathbf{x}; \mathbf{x}') = \frac{1}{\sqrt{s^2 + 2ss'\cos(\phi - \phi') + s'^2 + (z - z')^2}} + F(\mathbf{x}; \mathbf{x}'), \tag{2.7-2}$$

it is evident through my method of images argument that $F(\mathbf{x}; \mathbf{x}')$ is $(-1) \times$ the first term with the additional replacement $z' \to -z'$:

$$\boxed{G(\mathbf{x}; \mathbf{x}') = \frac{1}{\sqrt{s^2 + 2ss'\cos(\phi - \phi') + s'^2 + (z - z')^2}} - \frac{1}{\sqrt{s^2 + 2ss'\cos(\phi - \phi') + s'^2 + (z + z')^2}}} \tag{2.7-3}$$

which has the property of vanishing at the surface $z = 0$ and spatial infinity, while also keeping the condition that $\nabla'^2 F(\mathbf{x}; \mathbf{x}') = 0$.

(b) If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (s, ϕ, z).

Solution: The potential for Dirichlet boundary conditions can be simplified through the assumption $\rho = 0$, and the use of $\hat{n}' = -\hat{z}'$ (outwards from half-space $z \geq 0$):

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\cancel{\int_V d^3x' \rho(\mathbf{x}') G(\mathbf{x}; \mathbf{x}')} + \frac{1}{4\pi}\oint_S da' \phi(\mathbf{x}') \frac{\partial G}{\partial z'} \\ &= \frac{1}{4\pi}\oint_S da' \phi(\mathbf{x}') \frac{\partial G}{\partial z'},\end{aligned} \tag{2.7-4}$$

The normal derivative is simple to find,

$$\begin{aligned}\frac{\partial G}{\partial z'} &= \lim_{z' \to 0}\left[\frac{z - z'}{(s^2 + 2ss'\cos(\phi - \phi') + s'^2 + (z - z')^2)^{3/2}} + \frac{z + z'}{(s^2 + 2ss'\cos(\phi - \phi') + s'^2 + (z + z')^2)^{3/2}}\right] \\ &= \frac{2z}{(s^2 + 2ss'\cos(\phi - \phi') + s'^2 + z^2)^{3/2}},\end{aligned} \tag{2.7-5}$$

such that after using $da' = s'ds'd\phi'$ I find

$$\boxed{\Phi(\mathbf{x}) = \frac{V}{4\pi}\int_0^{2\pi} d\phi' \int_0^a ds' \frac{2zs'}{(s^2 + 2ss'\cos(\phi - \phi') + s'^2 + z^2)^{3/2}}} \tag{2.7-6}$$

as the potential at some point s, ϕ, z in cylindrical coordinates. Note that since the integration runs only on $s \in \{0, a\}$ I can make the replacement $\phi(\mathbf{x}') = V$.

(c) Show that, along the axis of the circle ($s = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right)$$

Solution: Take the case where $s = 0$,

$$\Phi(\mathbf{x}) = \frac{Vz}{2\pi}\int_0^{2\pi} d\phi \int_0^a ds' \frac{s'}{(s'^2 + z^2)^{3/2}}, \tag{2.7-7}$$

the integral is elementary after substituting $u = s'^2 + z^2$,

$$\Phi(\mathbf{x}) = \frac{Vz}{2}\int_{z^2}^{a^2+z^2} du\, u^{-3/2} \iff \boxed{\Phi(z) = V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right)} \tag{2.7-8}$$

the potential depends only on z due to the rotational symmetry under the choice $s = 0$.

(d) Show that at large distances ($s^2 + z^2 \gg a^2$) the potential can be expanded in a power series in $(s^2 + z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{Va^2}{2}\frac{z}{(s^2 + z^2)^{3/2}}\left[1 - \frac{3a^2}{4(s^2 + z^2)} + \frac{5(3s^2a^2 + a^4)}{8(s^2 + z^2)^2} + \cdots\right].$$

Verify that the results of parts c and d are consistent with each other in their common range of validity.

Solution: To get the overall factor in the problem statement I need to pull out a factor of $s^2 + z^2$, which is supported by the fact that it allows me to (eventually) perform a binomial expansion of the integrand (2.7-6):

$$\Phi(\mathbf{x}) = \frac{Vz}{2\pi}\frac{1}{(s^2 + z^2)^{3/2}}\int_0^{2\pi} d\phi \int_0^a ds' s' \left(1 + \frac{s'^2 + 2ss'\cos(\phi - \phi')}{s^2 + z^2}\right)^{-3/2} \tag{2.7-9}$$

To justify the expansion, the angular term is bounded between ± 1 and is small, and the integration s' is over a region $[0, a]$ which is small under the condition $s^2 + z^2 \gg a^2$, hence $s^2 + z^2 \gg s'^2$. The only issue is the fact that the angular term involves a factor of s in the numerator, but I will assume that z is large enough in itself to compensate for this factor (so that $s^2 + z^2 \gg s$). Since $(1 + x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \cdots$,

$$\begin{aligned}
\Phi(\mathbf{x}) = \frac{Vz}{2\pi}\frac{1}{(s^2 + z^2)^{3/2}}\int_0^{2\pi} d\phi \int_0^a ds' \Big(s' - \frac{3}{2}\frac{1}{s^2 + z^2}\left(s'^3 + 2ss'^2\cancel{\cos(\phi - \phi')}\right) \\
+ \frac{15}{8}\frac{1}{(s^2 + z^2)^2}\left(s'^5 + 4s^2s'^3\cos^2(\phi - \phi') + 4ss'^3\cancel{\cos(\phi - \phi')}\right) + \cdots\Big) \\
= \frac{V}{2\pi}\frac{z}{(s^2 + z^2)^{3/2}}\left(\pi a^2 - \frac{3}{2}\left(\frac{\pi}{2}a^4\right)\frac{1}{s^2 + z^2} + \frac{15}{8}\frac{a^4/3 + a^2s^2}{(s^2 + z^2)^2} + \cdots\right),
\end{aligned} \tag{2.7-10}$$

which can be rewritten as

$$\boxed{\Phi(s, z) = \frac{Va^2}{2}\frac{z}{(s^2 + z^2)^{3/2}}\left[1 - \frac{3a^2}{4(s^2 + z^2)} + \frac{5(3s^2a^2 + a^4)}{8(s^2 + z^2)^2} + \cdots\right]} \tag{2.7-11}$$

as desired. Relation (2.7-11) and (2.7-8) are common for the choice of $s = 0$, under which follows that $z \gg a$ and (2.7-11) becomes

$$\begin{aligned}
\Phi(z) = \frac{V}{2}\left(\frac{a}{z}\right)^2\left[1 - \frac{3}{4}\left(\frac{a}{z}\right)^2 + \frac{5}{8}\left(\frac{a}{z}\right)^4 + \cdots\right] = \frac{V}{2}\left[\left(\frac{a}{z}\right)^2 - \frac{3}{4}\left(\frac{a}{z}\right)^4 + \cdots\right] \\
= V\left[1 - \left(1 - \frac{1}{2}\left(\frac{a}{z}\right)^2 + \frac{3}{8}\left(\frac{a}{z}\right)^4 + \cdots\right)\right]
\end{aligned} \tag{2.7-12}$$

Note that

$$\frac{z}{\sqrt{a^2+z^2}} = \left(1+\left(\frac{a}{z}\right)^2\right)^{-1/2} = 1-\frac{1}{2}\left(\frac{a}{z}\right)^2+\frac{3}{8}\left(\frac{a}{z}\right)^4, \tag{2.7-13}$$

which allows me to rewrite (2.7-12) in the form

$$\boxed{\Phi(z) = V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)} \tag{2.7-14}$$

which is (2.7-8).

Jackson (2.15)

This question has two parts:

(a) Show that the Green function $G(x,y;x',y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0\le x\le 1$, $0\le y\le 1$, has an expansion

$$G(x,y;x',y') = 2\sum_{n\ge 1} g_n(y,y')\sin(n\pi x)\sin(n\pi x'),$$

where $g_n(y,y')$ satisfies

$$\left(\frac{\partial^2}{\partial y'^2}-n^2\pi^2\right)g_n(y,y') = -4\pi\delta(y'-y), \quad \text{and} \quad g_n(y,0)=g_n(y,1)=0$$

Solution: This Green's function is a function of 4 coordinates, and a Fourier series expansion must account for each of them. For fixed ends (Dirichlet) the mode functions are

$$u_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}(x-a)\right) \xrightarrow{L=1,\ a=0} \sqrt{2}\sin(n\pi x), \tag{2.15-1}$$

where I used $L=1, a=0$ for the given square region. In solving Laplace's equation in 2d via separation of variables, we are forced to choose one function as oscillatory and the other is exponential. I choose to expand in terms of sines for x, x', and to leave the y, y' dependence as a function $g_n(y,y')$, giving a general expansion

$$\begin{aligned} G(x,y;x',y') &= \sum_{n\ge 1} g_n(y,y')u_n(x)u_n(x') \\ \iff &\boxed{G(x,y;x',y') = 2\sum_{n\ge 1} g_n(y,y')\sin(n\pi x)\sin(n\pi x')} \end{aligned} \tag{2.15-2}$$

which agrees with the problem statement. To find the equation obeyed by g_n, I will take inspiration from the defining property of a 2D Green's function,

$$\nabla'^2 G(x,y;x',y') = -4\pi\delta(x-x')\delta(y-y'), \tag{2.15-3}$$

which can be reduced to an equation on g_n by substitution of (2.15-2):

$$\begin{aligned} -4\pi\delta(x-x')\delta(y-y') &= 2\sum_{n\ge 1}\nabla'^2\left[g_n(y,y')\sin(n\pi x')\right]\sin(n\pi x) \\ &= 2\sum_{n\ge 1}\left(\frac{\partial^2}{\partial y'^2}-(n\pi)^2\right)g_n(y,y')\sin(n\pi x)\sin(n\pi x'). \end{aligned} \tag{2.15-4}$$

To simplify this relation I will exploit the orthogonality of the mode functions,

$$\int_0^1 dx\,\sin(n\pi x)\sin(m\pi x) = \frac{1}{2}\delta_{nm}, \tag{2.15-5}$$

through a multiplication of a ratio of sines $\sin(m\pi x)/\sin(m\pi x')$ followed by an integration over x,

$$\begin{aligned}
-4\pi\delta(y-y')\int_0^1 dx\ \delta(x-x')\frac{\sin(m\pi x)}{\sin(m\pi x')} &= 2\sum_{n\geq 1}\left(\frac{\partial^2}{\partial y'^2}-(n\pi)^2\right)g_n(y,y')\frac{\sin(n\pi x')}{\sin(m\pi x')} \\
&\qquad\times\int_0^1 dx\ \sin(n\pi x)\sin(m\pi x) \\
-4\pi\delta(y-y') &= 2\sum_{n\geq 1}\left(\frac{\partial^2}{\partial y'^2}-(n\pi)^2\right)g_n(y,y')\frac{\sin(n\pi x')}{\sin(m\pi x')}\left(\frac{1}{2}\delta_{nm}\right),
\end{aligned} \tag{2.15-6}$$

$$\therefore\ \boxed{\left(\frac{\partial^2}{\partial y'^2}-n^2\pi^2\right)g_n(y,y') = -4\pi\delta(y'-y)} \tag{2.15-7}$$

No second-order differential equation is complete without two boundary conditions. For Dirichlet boundary conditions, I require $g_n(y,y')$ to vanish $y'=0,1\ \forall y$, such that $\boxed{g_n(y,0)=g_n(y,1)=0}$ are the boundary conditions on $g_n(y,y')$.

(b) Taking for $g_n(y,y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions, $y'<y$ and $y'>y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G(x,x';y,y') = 8\sum_{n\geq 1}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\sinh[n\pi(1-y_>)]$$

where $y_<(y_>)$ is the smaller(larger) of y and y'.

Solution: I will analyze (2.15-7) in two cases,

$$\begin{aligned}
(y>y') &\quad \frac{\partial^2 g_n}{\partial y'^2} = n^2\pi^2 g_n \implies g_n(y,y') = A\sinh(n\pi y'), \\
(y<y') &\quad \frac{\partial^2 g_n}{\partial y'^2} = n^2\pi^2 g_n \implies g_n(y,y') = B\sinh(n\pi(1-y')).
\end{aligned} \tag{2.15-8}$$

I chose sinh to enforce the fixed ends boundary conditions $g_n(y,0)=g_n(y,1)=0$. I can write the general solution with the use of Heaviside functions,

$$g_n(y,y') = \theta(y-y')A\sinh(n\pi y') + \theta(y'-y)B\sinh(n\pi(1-y')). \tag{2.15-9}$$

The trick in determining the coefficients is to substitute (2.15-9) into (2.15-7), where

$$\begin{aligned}
\frac{\partial g_n}{\partial y'} &= \theta(y-y')n\pi A\cosh(n\pi y') - \theta(y'-y)n\pi B\cosh(n\pi(1-y')) \\
&\qquad + \delta(y-y')\left(-A\sinh(n\pi y)+B\sinh(n\pi(1-y))\right) \\
\frac{\partial^2 g_n}{\partial y'^2} &= n^2\pi^2 g_n + n\pi\delta(y-y')\left(-A\cosh(n\pi y)+B\cosh(n\pi(1-y))\right) \\
&\qquad + \delta'(y-y')\left(-A\sinh(n\pi y)+B\sinh(n\pi(1-y))\right)
\end{aligned} \tag{2.15-10}$$

For (2.15-10) to match (2.15-7), I find two sets of equations which can be written in matrix form,

$$\begin{pmatrix}\cosh(n\pi y) & \cosh(n\pi(1-y)) \\ -\sinh(n\pi y) & \sinh(n\pi(1-y))\end{pmatrix}\begin{pmatrix}A \\ B\end{pmatrix} = \begin{pmatrix}4/n \\ 0\end{pmatrix}. \tag{2.15-11}$$

Since the Wronksian can be evaluated as

$$\begin{aligned}
\cosh(n\pi y)\sinh(n\pi(1-y)) + \cosh(n\pi(1-y))\sinh(n\pi y) &= \sinh(n\pi y + n\pi - n\pi y) \\
&= \sinh(n\pi),
\end{aligned} \tag{2.15-12}$$

(2.15-11) can be easily inverted:

$$\begin{aligned}\begin{pmatrix}A\\B\end{pmatrix} &= \frac{1}{\sinh(n\pi)}\begin{pmatrix}\sinh(n\pi(1-y)) & -\cosh(n\pi(1-y))\\ \sinh(n\pi y) & \cosh(n\pi y)\end{pmatrix}\begin{pmatrix}4/n\\0\end{pmatrix}\\ &= \frac{4}{n\sinh(n\pi)}\begin{pmatrix}\sinh(n\pi(1-y))\\ \sinh(n\pi y)\end{pmatrix}.\end{aligned} \tag{2.15-13}$$

With A, B known, (2.15-9) can be written as

$$\begin{aligned}g_n(y,y') &= \theta(y-y')\left(\frac{4\sinh(n\pi y')\sinh(n\pi(1-y))}{n\sinh(n\pi)}\right)+\theta(y'-y)\left(\frac{4\sinh(n\pi y)\sinh(n\pi(1-y'))}{n\sinh(n\pi)}\right)\\ &= \frac{4\sinh(n\pi y_<)\sinh(n\pi(1-y_>))}{n\sinh(n\pi)},\end{aligned} \tag{2.15-14}$$

where $y_<(y_>)$ are the lesser(larger) of y, y'. With g_n determined, I only need to substitute it into (2.15-2) to get the answer,

$$\boxed{G(x,x';y,y') = 8\sum_{n\geq 1}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\sinh[n\pi(1-y_>)]} \tag{2.15-15}$$

which is the expansion of the Green's function in a 2D square with Dirichlet boundary conditions.

Chapter 3

Jackson (3.2)

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

(a) Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta), \tag{3.2-1}$$

where, for $l = 0$, $P_{l-1}(\cos\alpha) = -1$. What is the potential outside?

Solution: I will express the potential as an integral over the charge density of this distribution, then use the expansion of $1/R$:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{||\mathbf{r}-\mathbf{r}'||}\\ &= \frac{Q}{4\pi\epsilon_0}\frac{1}{4\pi R^2}\int_0^{2\pi}d\phi'\int_{-1}^{1}d(\cos\theta')\Theta(\theta'-\alpha)\int_0^\infty dr'\delta(r'-R)r'^2\frac{1}{||\mathbf{r}-\mathbf{r}'||}\\ &= \frac{Q}{4\pi\epsilon_0}\frac{1}{4\pi R^2}\sum_{l\geq 0}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)\int_0^{2\pi}d\phi'\int_{-1}^{1}d(\cos\theta')\Theta(\theta'-\alpha)Y^*_{lm}(\theta',\phi')\int_0^\infty dr'\delta(r'-R)r'^2\frac{r_<^l}{r_>^{l+1}},\end{aligned} \tag{3.2-2}$$

where since

$$\begin{aligned}\int_0^{2\pi}d\phi' Y^*_{lm}(\theta',\phi') &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta')\int_0^{2\pi}d\phi' e^{im\phi'}\\ &= \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta')2\pi\delta_{m,0},\end{aligned} \tag{3.2-3}$$

then (3.2-2) becomes

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{Q}{4\pi\epsilon_0}\frac{1}{2}\sum_{l\geq 0}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)\int_{-1}^{1}d(\cos\theta')\Theta(\cos\alpha-\cos\theta')P_l(\cos\theta')\\ &= \frac{Q}{8\pi\epsilon_0}\sum_{l\geq 0}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)\int_{-1}^{\cos\alpha}d(\cos\theta')P_l(\cos\theta')\end{aligned} \tag{3.2-4}$$

Note that I made the substitution $\Theta(\theta'-\alpha) = \Theta(\cos\alpha - \cos\theta')$, after which changing the integral bounds. Note that before the delta function, $r_<, r_> \in \{r, r'\}$, but now $r_<, r_> \in \{r, R\}$. My idea is to take investigate the $\theta = 0$ case, whereby using (Jackson 3.28),

$$P_l(x) = \frac{1}{2l+1}\frac{d}{dx}\left(P_{l+1}(x) - P_{l-1}(x)\right), \tag{3.2-5}$$

I can express the integrand of (3.2-4) as a total derivative and use the 1st FTC:

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{Q}{8\pi\epsilon_0}\sum_{l\geq 0}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\int_{-1}^{\cos\alpha}d(\cos\theta')\frac{d}{d(\cos\theta')}\left(P_{l+1}(\cos\theta') - P_{l-1}(\cos\theta')\right)\\ &= \frac{Q}{8\pi\epsilon_0}\sum_{l\geq 0}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left[P_{l+1}(\cos\theta) - P_{l-1}(\cos\theta)\right]\Big|_{-1}^{\cos\alpha}\\ &= \frac{Q}{8\pi\epsilon_0}\sum_{l\geq 0}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]\end{aligned} \tag{3.2-6}$$

where I made the argument that

$$P_l(-1) = (-1)^l P_l(1) = (-1)^l \implies P_{l+1}(-1) = P_{l-1}(-1) = -(-1)^l, \tag{3.2-7}$$

which sets the lower bound to zero. The potential for any θ is can be found by multiplying each term in the series (3.2-6) by a corresponding $P_l(\cos\theta)$,

$$\boxed{\Phi(\mathbf{r}) = \frac{Q}{8\pi\epsilon_0}\sum_{l\geq 0}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)} \tag{3.2-8}$$

For the potential inside the sphere, simply set $r_< = r$ and $r_> = R$, which coincides precisely with the problem statement (3.2-1). The potential outside can be found by setting $r_< = R$ and $r_> = r$.

(b) Find the magnitude and the direction of the electric field at the origin.

Solution: To find the electric field at the origin I need to take the gradient of Φ under the choice $r_< = r,\ r_> = R$ then evaluate it at $r = 0$. Clearly, for $l \geq 2$ then E_{in} is an explicit function of r and vanishes at the origin. The $l = 0$ term gives a constant, as $P_0(\cos\theta) = 1,\ r^0 = 1$, so it vanishes from the gradient. Thus, I only need to look at the $l = 1$ term of (3.2-8):

$$\begin{aligned}\Phi_{\text{in}}(r,\theta)\delta_{l,1} &= \frac{Q}{8\pi\epsilon_0 R^2}\frac{1}{3}\left(P_2(\cos\alpha) - P_0(\cos\alpha)\right) r P_1(\cos\theta) \\ &= \frac{Q}{8\pi\epsilon_0 R^2}\frac{1}{3}\left(\frac{3}{2}\cos^2\alpha - \frac{3}{2}\right) r\cos\theta \\ &= -\frac{Q}{16\pi\epsilon_0 R^2}\sin^2\alpha\,(r\cos\theta)\,.\end{aligned} \tag{3.2-9}$$

Now, note that since $z = r\cos\theta$, then $\nabla(r\cos\theta) = \nabla(z) = \hat{z}$, such that the corresponding electric field is

$$\boxed{\mathbf{E}_{\text{origin}} = -\nabla(\Phi_{\text{in}}(r,\theta))\Big|_{r=0} = \frac{Q}{16\pi\epsilon_0 R^2}\sin^2\alpha\hat{z}} \tag{3.2-10}$$

which for given α is a constant and in the $\hat{z}$ direction. Note that I suppressed the $\delta_{l,1}$ from (3.2-9) in (3.2-10) since I know the gradient (and $r = 0$) will omit the other l's naturally. The angular dependence has vanished by spherical symmetry, and is buried within the expression for $\hat{z}$ in terms of $\hat{r}, \hat{\theta}$.

(c) Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes (1) very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

Solution: I will start with the electric field for both cases since it is simple. For a small cap, $\alpha \sim 0$ and I am free to take $\sin^2\alpha \sim \alpha^2$ such that

$$\boxed{\mathbf{E}_{\text{origin}} \sim \frac{Q}{16\pi\epsilon_0 R^2}\alpha^2 \xrightarrow{\alpha\to 0} 0, \qquad (\alpha \sim 0)} \tag{3.2-11}$$

which vanishes for $\alpha = 0$ ... obvious through Gauss' Law since the charge is uniformly distributed across the surface for that case.

In the limit of a small cap at the south pole $\alpha \sim \pi$, I can approximate $\sin^2(\pi - \alpha) \sim (\pi - \alpha)^2$ such that

$$\boxed{\mathbf{E}_{\text{origin}} \sim \frac{Q}{16\pi\epsilon_0 R^2}(\pi - \alpha)^2 \xrightarrow{\alpha\to\pi} 0, \qquad (\alpha \sim \pi)} \tag{3.2-12}$$

which clearly vanishes under $\alpha = \pi$ because that case corresponds to no charge at all!

I will now look at the potential (3.2-8). For $\alpha \sim 0$ I will expand $\cos\alpha \sim 1 - \frac{\alpha^2}{2}$. Since I would have $P_l(\cos\alpha) \sim P_l\left(1 - \frac{\alpha^2}{2}\right)$ for $\alpha \ll 1$, I can Taylor expand the Legendre polynomial to first-order:

$$P_l\left(1 - \frac{\alpha^2}{2}\right) \sim P_l(1) + \frac{dP_l}{d\alpha}(1) \times \left(-\frac{\alpha^2}{2}\right), \tag{3.2-13}$$

thereby implying that

$$\begin{aligned} P_{l+1}\left(1-\frac{\alpha^2}{2}\right) - P_{l-1}\left(1-\frac{\alpha^2}{2}\right) &\sim (P_{l+1}(1) - P_{l-1}(1)) - \frac{\alpha^2}{2}\left(\frac{dP_{l+1}}{d\alpha}(1) - \frac{dP_{l-1}}{d\alpha}(1)\right) \\ &\sim (P_{l+1}(1) - P_{l-1}(1)) - (2l+1)\frac{\alpha^2}{2}, \end{aligned} \tag{3.2-14}$$

where I used (3.2-5) to simplify things (with $P_l(1) = 1$). For $l > 0$, the term in parentheses vanishes because the Legendre polynomials are normalized (and here the indices are ≥ 0), giving

$$P_{l+1}\left(1-\frac{\alpha^2}{2}\right) - P_{l-1}\left(1-\frac{\alpha^2}{2}\right) \sim -(2l+1)\frac{\alpha^2}{2}, \qquad (l > 0). \tag{3.2-15}$$

In the case $l = 0$ the first term doesn't vanish, as $P_{-1}(1) = -1$ from (3.2-7) which means the term in parentheses is 2:

$$P_1\left(1-\frac{\alpha^2}{2}\right) - P_{-1}\left(1-\frac{\alpha^2}{2}\right) \sim 2 - \frac{\alpha^2}{2}, \qquad (l = 0). \tag{3.2-16}$$

At last, I can expand (3.2-8) after isolating the $l = 0$ component from the rest:

$$\begin{aligned} \Phi(r,\theta) &= \frac{Q}{8\pi\epsilon_0}\left(P_1(\cos\alpha) - P_{-1}(\cos\alpha)\right)\frac{1}{r_>}P_0(\cos\theta) \\ &\qquad + \frac{Q}{8\pi\epsilon_0}\sum_{l=1}^{\infty}\frac{1}{2l+1}\left(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right)\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) \\ &\sim \frac{Q}{8\pi\epsilon_0 r_>}\left(2 - \frac{\alpha^2}{2}\right) - \frac{Q}{8\pi\epsilon_0}\frac{\alpha^2}{2}\sum_{l=1}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) \\ &\sim \frac{Q}{4\pi\epsilon_0 r_>} - \frac{Q}{8\pi\epsilon_0}\frac{\alpha^2}{2}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta), \end{aligned} \tag{3.2-17}$$

where I shifted my index in the last equality to absorb the $\alpha^2/2$ term. Elegantly, I can use (Jackson 3.38) to simplify the sum! Since $\cos\theta = \hat{r}\cdot\hat{z}$ and the charge giving rise to the potential is at $\mathbf{r}' = R\hat{z}$, the sum can be interpretted as $||\mathbf{r} - R\hat{z}||^{-1}$, further motivated by my choice $(r_<, r_>) \in \{r, R\}$:

$$\boxed{\Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r_>} - \frac{Q\alpha^2}{16\pi\epsilon_0}\frac{1}{||\mathbf{r} - R\hat{z}||} \qquad (\alpha \sim 0)} \tag{3.2-18}$$

which is the expansion of the potential to first order in the small parameter $\alpha \sim 0$. In the case $\alpha = 0$ only the first term remains, and for Φ_{out} it is simply the potential of a point charge Q centered at the origin. The second term is the potential of a point charge $-Q\alpha^2/4$ centered at $\mathbf{r} = R\hat{z}$ (since $\alpha \sim 0$), and is the term arising from the (small, $\alpha \sim 0$) cap that effectively becomes a point source for small α (smearing a negative charge over positive charge has the effects of an absence of charge). Note that the electric field (3.2-11) follows automatically from (3.2-18) from taking the gradient (under $r_> = R$ of course).

For the other limit, note that $\cos\alpha \approx -(1-(\pi-\alpha)^2/2)$ for $\alpha \sim \pi$. Under this approximation, the Legendre polynomial become

$$\begin{aligned} P_l(\cos\alpha) &\approx (-1)^l P_l\left(1 - \frac{(\pi-\alpha)^2}{2}\right) \\ &\approx (-1)^l P_l(1) - (-1)^l\frac{(\pi-\alpha)^2}{2}\frac{dP}{d\alpha}(1) \\ &= (-1)^l - (-1)^l\frac{(\pi-\alpha)^2}{2}\frac{dP}{d\alpha}(1) \end{aligned} \tag{3.2-19}$$

where I used the parity to pull out a factor of $(-1)^l$. Note that the $(-1)^l$ pre-factor accounts for the $l=-1$ case and I was free to assume the normalization $P_l(1)$. Therefore, I find that

$$\begin{aligned} P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) &= \cancel{\left[(-1)^{l+1} - (-1)^{l-1}\right]} + \frac{(\pi-\alpha)^2}{2}\left[\frac{dP_{l+1}}{d\alpha}(1) - \frac{dP_{l-1}}{d\alpha})(1)\right](-1)^l \\ &= \frac{2l+1}{2}(\pi-\alpha)^2(-1)^l \qquad \forall l \end{aligned} \tag{3.2-20}$$

Substituting this into (3.2-8), I find

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{Q}{16\pi\epsilon_0}(\pi-\alpha)^2\sum_{l\geq 0}(-1)^l\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) \\ &= \frac{Q}{16\pi\epsilon_0}(\pi-\alpha)^2\sum_{l\geq 0}\frac{r_<^l}{r_>^{l+1}}P_l(-\cos\theta) \end{aligned} \tag{3.2-21}$$

where I reabsorbed the parity factor into the Legendre polynomial. Using the fact that $\hat{r}\cdot(-\hat{z}) = -\cos\theta$, I can use (Jackson 3.38) to simply the sum:

$$\boxed{\Phi(\mathbf{r}) = \frac{Q(\pi-\alpha)^2}{16\pi\epsilon_0}\frac{1}{||\mathbf{r}+R\hat{z}||} \qquad (\alpha\sim\pi)} \tag{3.2-22}$$

This limiting behavior has a simple interpretation, as for $\alpha\sim\pi$ the only charge in the problem is very tiny "arc" of charge at $\mathbf{r}' = -R\hat{z}$, which for small α effectively forms the potential of a point charge $Q(\pi-\alpha)^2/4$. Note importantly that there is no extra term like in (3.2-18), as this limit does not correspond to an "almost full" spherical shell of charge, and the gradient of this limit (3.2-22) gives precisely the electric field (3.2-12) under $\alpha\sim\pi$ ... as it should!

Jackson (3.10)

For the cylinder in Problem 3.9 the cylindrical surface is made of two equal half-cylinders, one at potential V and the other at potential $-V$, so that

$$V(\phi, z) = \begin{cases} V, & -\pi/2 < \phi < \pi/2 \\ -V, & \pi/2 < \phi < 3\pi/2 \end{cases} \tag{3.10-1}$$

(a) Find the potential inside the cylinder.

Solution: Here we have that $\Phi \to 0$ on the top and bottom of the cylinder, implying that I can do a fixed ends mode expansion for the z-dimension:

$$Z(z) \propto \sin\left(\frac{n\pi}{L} z\right). \quad (n \in \mathbb{Z}^+) \tag{3.10-2}$$

where the $n = 0$ term is trivial but I will include it anyway. In ϕ we have periodic boundary conditions, so the mode expansion in that dimension is

$$\psi(\phi) \propto e^{imz}. \quad (m \in \mathbb{Z}) \tag{3.10-3}$$

Now for the radial dimension, I need to avoid the singularity at $s = 0$, and I will do so by picking the modified Bessel function of the first kind,

$$I_{|m|}\left(\frac{n\pi}{L} s\right), \tag{3.10-4}$$

which is zero at the origin. My reasoning for choosing the modified Bessel function is because I need it to behave like a hyperbolic trig function, as my other two dimensions are oscillatory. As is usual in the expansion of solutions Laplace's equation in cylindrical coordinates, the Bessel function index is the same as the azimuthal index, and the same wave number $n\pi/L$ as the z-dimension. With these mode functions, I will take a general expansion

$$\Phi_{\text{in.}}(s, \phi, z) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} \sin\left(\frac{n\pi z}{L}\right) I_{|m|}\left(\frac{n\pi s}{L}\right) e^{im\phi}. \tag{3.10-5}$$

All boundary conditions are satisfied except for the surfaces held at potentials $\pm V$, in which case I need to use orthogonality:

$$\begin{aligned} \Phi_{\text{in.}}(b, \phi, z) = V(\phi) &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} \sin\left(\frac{n\pi z}{L}\right) I_{|m|}\left(\frac{n\pi b}{L}\right) e^{im\phi} \\ \iff \int_0^{2\pi} d\phi\, V(\phi) e^{-ik\phi} \int_0^L dz \sin\left(\frac{l\pi z}{L}\right) &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} \int_0^L dz \sin\left(\frac{l\pi z}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \\ &\quad \times I_{|m|}\left(\frac{n\pi b}{L}\right) \int_0^{2\pi} d\phi\, e^{i(m-k)\phi} \\ A_{lk}\,(2\pi)\left(\frac{L}{2}\right) I_{|k|}\left(\frac{l\pi b}{L}\right) &= V\left(\int_{-\pi/2}^{\pi/2} d\phi e^{-ik\phi} - \int_{\pi/2}^{-\pi/2} d\phi e^{-ik\phi}\right)\left(-\frac{L}{l\pi}\cos\left(\frac{l\pi z}{L}\right)\right)\Bigg|_0^L \\ &= \frac{4VL}{\pi l k} \sin(\pi k/2)\,(1 - \cos(l\pi)), \end{aligned} \tag{3.10-6}$$

$$\therefore\ A_{lk} = \frac{4V}{\pi^2} \frac{\sin(\pi k/2)(1 - \cos(l\pi))}{lk} \frac{1}{I_{|k|}\left(\frac{l\pi b}{L}\right)}, \tag{3.10-7}$$

so I can express (3.10-5) as

$$\Phi_{\text{in.}}(s, \phi, z) = \frac{4V}{\pi^2} \sum_{l=0}^{\infty} \sum_{k=-\infty}^{\infty} \frac{\sin(\pi k/2)(1 - \cos(l\pi))}{lk} \sin\left(\frac{l\pi z}{L}\right) \frac{I_{|k|}\left(\frac{l\pi s}{L}\right)}{I_{|k|}\left(\frac{l\pi b}{L}\right)} e^{ik\phi}. \tag{3.10-8}$$

Using the fact that $\sin(\pi k/2) = 0$ if k is even and $1 - \cos(l\pi) = 0$ if l is even, I can restrict the sums to be only over odd integers. This motivates the substitutions $l \to 2l+1$ and $k \to 2k+1$, such that

$$\begin{aligned}\Phi_{\text{in.}}(s,\phi,z) &= \frac{4V}{\pi^2}\sum_{l=0}^{\infty}\sum_{k=-\infty}^{\infty}\frac{\sin\left(\frac{2k+1}{2}\pi\right)\left(1-\cos\left((2l+1)\pi\right)\right)}{(2l+1)(2k+1)}\sin\left(\frac{(2l+1)\pi z}{L}\right)\frac{I_{|2k+1|}\left(\frac{(2l+1)\pi s}{L}\right)}{I_{|2k+1|}\left(\frac{(2l+1)\pi b}{L}\right)}e^{i(2k+1)\phi}\\ &= \frac{8V}{\pi^2}\sum_{l=0}^{\infty}\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{(2l+1)(2k+1)}\sin\left(\frac{(2l+1)\pi z}{L}\right)\frac{I_{|2k+1|}\left(\frac{(2l+1)\pi s}{L}\right)}{I_{|2k+1|}\left(\frac{(2l+1)\pi b}{L}\right)}e^{i(2k+1)\phi}\end{aligned} \tag{3.10-9}$$

where I used the fact that $\sin((2k+1)\pi/2) = (-1)^k$ and $1 - \cos((2l+1)\pi) = 2$ when k, l are integers. Making the substitution $\sum_{k=-\infty}^{\infty} = 2\sum_{k=0}^{\infty}$, my solution for the potential inside the cylinder is

$$\boxed{\Phi_{\text{in.}}(s,\phi,z) = \frac{16V}{\pi^2}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2l+1)(2k+1)}\sin\left(\frac{(2l+1)\pi z}{L}\right)\frac{I_{|2k+1|}\left(\frac{(2l+1)\pi s}{L}\right)}{I_{|2k+1|}\left(\frac{(2l+1)\pi b}{L}\right)}e^{i(2k+1)\phi}} \tag{3.10-10}$$

(b) Assuming $L \gg b$, consider the potential at $z = L/2$ as a function of s and ϕ and compare it with two-dimensional Problem 2.13.

Solution: For $L \gg b$ I also have $L \gg s$ and it is valid to expand both Bessel functions for small arguments,

$$\frac{I_{|2k+1|}\left(\frac{(2l+1)\pi s}{L}\right)}{I_{|2k+1|}\left(\frac{(2l+1)\pi b}{L}\right)} = \frac{1}{\Gamma(2k+2)}\left(\frac{(2l+1)\pi s}{2L}\right)^{2k+1} \times \Gamma(2k+2)\left(\frac{2L}{(2l+1)\pi b}\right)^{2k+1} = \left(\frac{s}{b}\right)^{2k+1}, \tag{3.10-11}$$

Using this expansion and letting $z = L/2$, (3.10-10) can be written as

$$\Phi_{\text{in.}}\left(s,\phi,\frac{L}{2}\right) = \frac{16V}{\pi^2}\sum_{l=0}^{\infty}\frac{\sin\left(\frac{(2l+1)\pi}{2}\right)}{2l+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\left(\frac{se^{i\phi}}{b}\right)^{2k+1} = \frac{16V}{\pi^2}\tan^{-1}\left(\frac{s}{b}e^{i\phi}\right)\sum_{l=0}^{\infty}\frac{\sin\left(\frac{(2l+1)\pi}{2}\right)}{2l+1} \tag{3.10-12}$$

where I used the series expansion of arctangent. Using the fact that $\sin((2l+1)\pi/2) = (-1)^l$, I can write the sum over l as a variant on the alternating harmonic series,

$$\sum_{l=0}^{\infty}\frac{\sin\left(\frac{(2l+1)\pi}{2}\right)}{2l+1} = \sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1} = \frac{\pi}{4}, \tag{3.10-13}$$

giving

$$\Phi_{\text{in.}}\left(s,\phi,\frac{L}{2}\right) = \frac{4V}{\pi}\tan^{-1}\left(\frac{s}{b}e^{i\phi}\right). \tag{3.10-14}$$

The actual potential for this problem is the real part of (3.10-14), and I cannot just simply replace the exponential with a cosine. Since the real part of a complex number satisfies $Re[z] = \frac{1}{2}(z + z^*)$, the real part of the potential has the form

$$\Phi_{\text{in.}}\left(s,\phi,\frac{L}{2}\right) = \frac{2V}{\pi}\left(\tan^{-1}\left(\frac{s}{b}e^{i\phi}\right) + \tan^{-1}\left(\frac{s}{b}e^{-i\phi}\right)\right) = \frac{2V}{\pi}\tan^{-1}\left(\frac{\frac{s}{b}\left(e^{i\phi}+e^{-i\phi}\right)}{1-\frac{s^2}{b^2}}\right), \tag{3.10-15}$$

$$\therefore \boxed{\Phi_{\text{in.}}\left(s,\phi,\frac{L}{2}\right) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2sb\cos(\phi)}{b^2-s^2}\right)} \tag{3.10-16}$$

where I used the addition formula for two arctangents. This is the potential (3.10-10) at $z = L/2$ for $L \gg b$. Comparing this to (Jackson 2.13),

$$\Phi = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2sb\cos\phi}{b^2-s^2}\right), \tag{3.10-17}$$

I find that the two are consistent under the choice $V_1 = V$ and $V_2 = -V$, as given in this problem.

Problem (3.23)

A point charge q is located at the point (ρ', ϕ', z') inside a grounded cylindrical box defined by the surfaces $z = 0, z = L, \rho = a$. Show that the potential inside the box can be express in the following alternating forms:

$$\begin{aligned}
\Phi(\mathbf{x}, \mathbf{x}') &= \frac{q}{\pi\epsilon_0 a} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{e^{im(\phi-\phi')} J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{x_{mn} J_{m+1}^2(x_{mn}) \sinh\left(\frac{x_{mn}L}{a}\right)} \sinh\left[\frac{x_{mn}}{a} z_<\right] \sinh\left[\frac{x_{mn}}{a}(L - z_>)\right] \\
\Phi(\mathbf{x}, \mathbf{x}') &= \frac{q}{\pi\epsilon_0 L} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) \frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)} \\
&\qquad \times \left[I_m\left(\frac{n\pi a}{L}\right) K_m\left(\frac{n\pi\rho_>}{L}\right) - K_m\left(\frac{n\pi a}{L}\right) I_m\left(\frac{n\pi\rho_>}{L}\right)\right] \\
\Phi(\mathbf{x}, \mathbf{x}') &= \frac{2q}{\pi\epsilon_0 L a^2} \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{im(\phi-\phi')} \sin\left(\frac{k\pi z}{L}\right) \sin\left(\frac{k\pi z'}{L}\right) J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2\right] J_{m+1}^2(x_{mn})}
\end{aligned} \tag{3.23-1}$$

Discuss the relation of the last expansion (with its extra summation) to the other two.

Solution: To get the first potential I will expand the radial modes as Bessel functions,

$$J_m\left(\frac{x_{mn}}{a}\rho\right) J_m\left(\frac{x_{mn}}{a}\rho'\right) \tag{3.23-2}$$

where I chose x_{mn} to be dimensionless and so the boundary condition at $\rho, \rho' = a$ is zero (x_{mn} is the root of J). With this choice, I will let the z-dependence be exponential:

$$\begin{aligned}
\mathcal{Z}(z, z') = \theta(z' - z) A \sinh\left(\frac{x_{mn}}{a} z\right) \sinh\left(\frac{x_{mn}}{a}(L - z')\right) \\
+ \theta(z - z') B \sinh\left(\frac{x_{mn}}{a}(L - z)\right) \sinh\left(\frac{x_{mn}}{a} z'\right)
\end{aligned} \tag{3.23-3}$$

Where I took the same wave-number as (3.23-2) and chose the sinh's to enforce the vanishing boundary conditions at $z = 0, L$. Lastly, we have periodic boundary conditions in ϕ (like always), so I take the mode function $\exp(im(\phi - \phi'))$ (with same m index as the other oscillating function). Taking the product of these three functions and summing over all indices,

$$\begin{aligned}
\Phi(\mathbf{x}, \mathbf{x}') = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} e^{im(\phi-\phi')} J_m\left(\frac{x_{mn}}{a}\rho\right) J_m\left(\frac{x_{mn}}{a}\rho'\right) \Big[\theta(z' - z) A \sinh\left(\frac{x_{mn}}{a} z\right) \sinh\left(\frac{x_{mn}}{a}(L - z')\right) \\
+ \theta(z - z') B \sinh\left(\frac{x_{mn}}{a}(L - z)\right) \sinh\left(\frac{x_{mn}}{a} z'\right)\Big].
\end{aligned} \tag{3.23-4}$$

Using the fact that

$$\nabla^2 \Phi(\mathbf{x}, \mathbf{x}') = -\frac{q}{\epsilon_0} \frac{\delta(\rho - \rho')}{\rho} \delta(z - z') \delta(\phi - \phi'), \tag{3.23-5}$$

I will take the Laplacian of (3.23-4) to get

$$\begin{aligned}
\nabla^2\Phi(\mathbf{x},\mathbf{x}') &= \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left[-\frac{m^2}{s^2}+\frac{m^2}{s^2}-\left(\frac{x_{mn}}{a}\right)^2\right]e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}}{a}\rho\right)J_m\left(\frac{x_{mn}}{a}\rho'\right)\\
&\quad\times\left[\theta(z'-z)A\sinh\left(\frac{x_{mn}}{a}z\right)\sinh\left(\frac{x_{mn}}{a}(L-z')\right)+\theta(z-z')B\sinh\left(\frac{x_{mn}}{a}(L-z)\right)\sinh\left(\frac{x_{mn}}{a}z'\right)\right]\\
&\quad+\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}}{a}\rho\right)J_m\left(\frac{x_{mn}}{a}\rho'\right)\frac{\partial^2}{\partial z^2}\left[\theta(z'-z)A\sinh\left(\frac{x_{mn}}{a}z\right)\sinh\left(\frac{x_{mn}}{a}(L-z')\right)\right.\\
&\qquad\qquad\left.+\theta(z-z')B\sinh\left(\frac{x_{mn}}{a}(L-z)\right)\sinh\left(\frac{x_{mn}}{a}z'\right)\right]\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}}{a}\rho\right)J_m\left(\frac{x_{mn}}{a}\rho'\right)\\
&\quad\times\left\{-\frac{x_{nm}}{a}\delta(z-z')\left[A\cosh\left(\frac{x_{nm}}{a}z'\right)\sinh\left(\frac{x_{nm}}{a}(L-z')\right)+B\cosh\left(\frac{x_{nm}}{a}(L-z')\right)\sinh\left(\frac{x_{nm}}{a}z'\right)\right]\right.\\
&\quad\left.+\delta'(z-z')\left[-A\sinh\left(\frac{x_{mn}}{a}z'\right)\sinh\left(\frac{x_{mn}}{a}(L-z')\right)+B\sinh\left(\frac{x_{mn}}{a}(L-z')\right)\sinh\left(\frac{x_{mn}}{a}z'\right)\right]\right.\\
&= -\frac{q}{\epsilon_0}\frac{\delta(\rho-\rho')}{\rho}\delta(z-z')\delta(\phi-\phi').
\end{aligned}\tag{3.23-6}$$

In order for (3.23-6) to match (3.23-5), I require that $A=B$ so the δ' term cancels, leaving the condition

$$\begin{aligned}
-\frac{q}{\epsilon_0}\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi') &= -\frac{1}{a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}Ax_{mn}e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}}{a}\rho\right)J_m\left(\frac{x_{mn}}{a}\rho'\right)\\
&\quad\times\left[\cosh\left(\frac{x_{nm}}{a}z'\right)\sinh\left(\frac{x_{nm}}{a}(L-z')\right)+\cosh\left(\frac{x_{nm}}{a}(L-z')\right)\sinh\left(\frac{x_{nm}}{a}z'\right)\right]\\
&= -\frac{1}{a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}Ax_{mn}\sinh\left(\frac{x_{nm}}{a}L\right)e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}}{a}\rho\right)J_m\left(\frac{x_{mn}}{a}\rho'\right),
\end{aligned}\tag{3.23-7}$$

where I used the addition formula for hyperbolic functions. Using the orthogonality conditions

$$\int_0^a d\rho\rho J_m\left(\frac{x_{mn}}{a}\rho\right)J_m\left(\frac{x_{ml}}{a}\rho\right)=\frac{a^2}{2}J_{m+1}^2(x_{mn})\delta_{n,l},\qquad\qquad \int_0^{2\pi}d\phi e^{i(m+k)(\phi-\phi')}=2\pi\delta_{m,k},\tag{3.23-8}$$

I can rewrite (3.23-7) as

$$\begin{aligned}
\frac{q}{\epsilon_0} &= \frac{1}{a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}Ax_{mn}\sinh\left(\frac{x_{mn}}{a}L\right)(2\pi\delta_{km})\left(\frac{a^2}{2}J_{m+1}^2(x_{mn})\delta_{n,l}\right),\\
\iff A &= \frac{q}{\pi\epsilon_0 a}\frac{1}{x_{mn}J_{mn}^2(x_{mn})\sinh\left(\frac{x_{mn}}{a}L\right)}=B.
\end{aligned}\tag{3.23-9}$$

This can be substituted into (3.23-4) to give the solution

$$\begin{aligned}
\Phi(\mathbf{x},\mathbf{x}') &= \frac{q}{\pi\epsilon_0 a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{x_{mn}J_{m+1}^2(x_{mn})\sinh\left(\frac{x_{mn}L}{a}\right)}\\
&\quad\times\left[\theta(z'-z)\sinh\left(\frac{x_{mn}}{a}z\right)\sinh\left(\frac{x_{mn}}{a}(L-z')\right)\right.\\
&\qquad\left.+\theta(z-z')\sinh\left(\frac{x_{mn}}{a}(L-z)\right)\sinh\left(\frac{x_{mn}}{a}z'\right)\right]
\end{aligned}\tag{3.23-10}$$

which can be rewritten in the $z_<, z_>$ notation as

$$\boxed{\Phi(\mathbf{x},\mathbf{x}')=\frac{q}{\pi\epsilon_0 a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{x_{mn}J_{m+1}^2(x_{mn})\sinh\left(\frac{x_{mn}L}{a}\right)}\sinh\left[\frac{x_{mn}}{a}z_<\right]\sinh\left[\frac{x_{mn}}{a}(L-z_>)\right]}\tag{3.23-11}$$

which is the first representation of the potential.

To get the second representation of the potential, I will note the completeness relations

$$\delta(z-z') = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right), \qquad \delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}, \tag{3.23-12}$$

which allow me to make an expansion of the Green's function equation,

$$\nabla^2 G(\mathbf{x};\mathbf{x}') = \frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\left(-\frac{4\pi}{\rho}\delta(\rho-\rho')\right). \tag{3.23-13}$$

This form inspires me to take the Green's function choice

$$\begin{aligned} G(\mathbf{x};\mathbf{x}') &= \frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) g_{nm}\left(\rho;\rho'\right), \\ \iff \nabla^2 G(\mathbf{x};\mathbf{x}') &= \frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \\ &\qquad \times\left[-\left(\frac{m}{\rho}\right)^2 - \left(\frac{n\pi}{L}\right)^2 + \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)\right] g_{nm}\left(\rho;\rho'\right) \end{aligned} \tag{3.23-14}$$

which through a comparison to (3.23-13) implies that $g_{nm}(\rho;\rho')$ satisfies

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_{nm}}{d\rho}\right) - \left[\left(\frac{m}{\rho}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right] g_{nm} = -\frac{4\pi}{\rho}\delta(\rho-\rho'), \tag{3.23-15}$$

which means g_{nm} is a solution to the inhomogenous modified Bessel equation. I will first solve the homogenous equation then determine coefficients to match the discontinuity (3.23-15). g_{nm} must be a linear combination of I_m and K_m, behave at $\rho = 0$ and vanish at $\rho = a$, and be symmetric in ρ, ρ'. This justifies the form

$$g_{nm}(\rho;\rho') = AI_m\left(\frac{n\pi}{L}\rho_<\right) I_m\left(\frac{n\pi}{L}\rho_>\right) + BI_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right) \tag{3.23-16}$$

for some constants A, B fixed to match boundary conditions and the inhomogenous solution. Note that for $x \ll 1$ then $I_m(x)K_m(x) \sim$ const. and the ansatz behaves at the origin, which is the reasoning for why I omitted the K_mK_m term. My reasoning for taking pairs is to make it symmetric. In the case $\rho = a$, this ansatz must vanish,

$$0 = AI_m\left(\frac{n\pi}{L}\rho'\right) I_m\left(\frac{n\pi a}{L}\right) + BI_m\left(\frac{n\pi}{L}\rho'\right) K_m\left(\frac{n\pi a}{L}\right) \implies B = -A\frac{I_m\left(\frac{n\pi a}{L}\right)}{K_m\left(\frac{n\pi a}{L}\right)}, \tag{3.23-17}$$

such that (3.23-16) is determined to an overall coefficient,

$$g_{nm}(\rho;\rho') = AI_m\left(\frac{n\pi}{L}\rho_<\right)\left(I_m\left(\frac{n\pi}{L}\rho_>\right) - \frac{I_m\left(\frac{n\pi a}{L}\right)}{K_m\left(\frac{n\pi a}{L}\right)}K_m\left(\frac{n\pi}{L}\rho_>\right)\right). \tag{3.23-18}$$

I will now integrate infinitesimally around ρ' in (3.23-15),

$$\begin{aligned} \int_{\rho'-\epsilon}^{\rho'+\epsilon} d\rho\frac{d}{d\rho}\left(\rho\frac{dg_{nm}}{d\rho}\right) - \cancel{\int_{\rho'-\epsilon}^{\rho'+\epsilon} d\rho\,\rho\left[\left(\frac{m}{\rho}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right] g_{nm}} &= -4\pi\int_{\rho'-\epsilon}^{\rho'+\epsilon} d\rho\delta(\rho-\rho') \\ \rho\frac{dg_{nm}}{d\rho}\bigg|_{\rho'+\epsilon} - \rho\frac{dg_{nm}}{d\rho}\bigg|_{\rho'-\epsilon} &= -4\pi \\ \frac{dg_{nm}}{d\rho}\bigg|_{\rho'+\epsilon} - \frac{dg_{nm}}{d\rho}\bigg|_{\rho'-\epsilon} &= -\frac{4\pi}{\rho'} \end{aligned} \tag{3.23-19}$$

In the first derivative term, the evaluation forces $\rho_< = \rho'$ and $\rho_> = \rho$, while the second derivative term forces $\rho_< = \rho$ and $\rho_> = \rho'$. Substituting (3.23-18) into (3.23-19) using these choices, I find that the $I_m I_m$ terms mutually cancel, leaving just

$$\begin{aligned} -\frac{4\pi}{\rho'} &= -\frac{n\pi}{L} A \frac{I_m\left(\frac{n\pi a}{L}\right)}{K_m\left(\frac{n\pi a}{L}\right)} \left[I_m\left(\frac{n\pi}{L}\rho'\right) K_m'\left(\frac{n\pi}{L}\rho'\right) - I_m'\left(\frac{n\pi}{L}\rho'\right) K_m\left(\frac{n\pi}{L}\rho'\right)\right] \\ &= -\frac{n\pi}{L} A \frac{I_m\left(\frac{n\pi a}{L}\right)}{K_m\left(\frac{n\pi a}{L}\right)} W\left[I_m\left(\frac{n\pi}{L}\rho'\right), K_m\left(\frac{n\pi}{L}\rho'\right)\right], \end{aligned} \tag{3.23-20}$$

where $W[\cdots]$ is the Wronksian. Note that the overall factor of $n\pi/L$ came from the chain rule. Equation (Jackson 3.147) gives a helpful formula for this Wronksian,

$$W\left[I_m(x), K_m(x)\right] = -\frac{1}{x}, \tag{3.23-21}$$

so I can simplify (3.23-20) as

$$-\frac{4\pi}{\rho'} = \frac{A}{\rho'} \frac{I_m\left(\frac{n\pi a}{L}\right)}{K_m\left(\frac{n\pi a}{L}\right)} \implies A = -4\pi \frac{K_m\left(\frac{n\pi a}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}, \tag{3.23-22}$$

such that (3.23-18) can be written as

$$g_{nm}(\rho;\rho') = 4\pi \frac{I_m\left(\frac{n\pi}{L}\rho_<\right)}{I_m\left(\frac{n\pi a}{L}\right)} \left(I_m\left(\frac{n\pi a}{L}\right) K_m\left(\frac{n\pi}{L}\rho_>\right) - K_m\left(\frac{n\pi a}{L}\right) I_m\left(\frac{n\pi}{L}\rho_>\right)\right). \tag{3.23-23}$$

At last, I can substitute this form into (3.23-14) to get the potential,

$$\boxed{\begin{aligned} \Phi(\mathbf{x},\mathbf{x}') = \frac{q}{4\pi\epsilon_0} G(\mathbf{x},\mathbf{x}') = \frac{q}{\pi\epsilon_0 L} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) \frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)} \\ \times \left[I_m\left(\frac{n\pi a}{L}\right) K_m\left(\frac{n\pi\rho_>}{L}\right) - K_m\left(\frac{n\pi a}{L}\right) I_m\left(\frac{n\pi\rho_>}{L}\right)\right] \end{aligned}}$$

which is indeed the second representation of the potential (3.23-1).

For the third representation of the potential, I will perform an eigenfunction expansion. Solving Poisson's eigenvalue equation in cylindrical coordinates with $\psi = \Gamma(\rho)Q(\phi)Z(z)$, I will choose to let Q, Z oscillate,

$$\begin{aligned} Q = e^{im\phi} &\implies \frac{Q''}{Q} = -m^2, \\ Z = \sin\left(\frac{k\pi z}{L}\right) &\implies \frac{Z''}{Z} = -\left(\frac{k\pi}{L}\right)^2, \end{aligned} \tag{3.23-24}$$

so that the corresponding eigenvalue equation can be written as

$$\begin{aligned} \frac{(\rho\Gamma')'}{\rho\Gamma} + \frac{1}{\rho^2}\frac{Q''}{Q} + \frac{Z''}{Z} &= -\lambda \\ \Gamma'' + \frac{1}{\rho}\Gamma' + \left[\lambda - \left(\frac{m}{\rho}\right)^2 - \left(\frac{k\pi}{L}\right)^2\right]\Gamma &= 0 \end{aligned} \tag{3.23-25}$$

Note that the ϕ function has periodic boundary conditions (as it should), and the z function vanishes at the top and bottom of the cylinder. Note that (3.23-25) is just the Bessel equation, with solution $\Gamma(\rho) = J_m(q\rho) = J_m\left(\frac{x_{mn}}{a}\rho\right)$, under the choice

$$q = \sqrt{\lambda - \left(\frac{k\pi}{L}\right)^2} = \frac{x_{mn}}{a}, \implies \lambda_{kmn} = \left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2. \tag{3.23-26}$$

The eigenfunction ψ_{kmn} can now be expressed as

$$\psi_{kmn}(\rho,\phi,z) = Ae^{im\phi}\sin\left(\frac{k\pi z}{L}\right)J_m\left(\frac{x_{mn}}{a}\rho\right), \tag{3.23-27}$$

where A is a normalization constant. Normalizing,

$$\begin{aligned}1 = \int d^3x\ \psi^*_{kmn}\psi_{kmn} &= |A|^2\int_0^{2\pi} d\phi\int_0^L dz\ \sin^2\left(\frac{k\pi z}{L}\right)\int_0^a d\rho\ \rho J_m\left(\frac{x_{mn}}{a}\rho\right)J_m\left(\frac{x_{mn}}{a}\rho\right)\\ &= |A|^2\left(\frac{\pi La^2}{2}J^2_{m+1}(x_mn)\right) \implies A = \sqrt{\frac{2}{\pi La^2}}\frac{1}{J_{m+1}(x_{mn})},\end{aligned} \tag{3.23-28}$$

which fully determines the eigenfunctions for the problem:

$$\psi_{kmn}(\rho,\phi,z) = \sqrt{\frac{2}{\pi La^2}}e^{im\phi}\sin\left(\frac{k\pi z}{L}\right)\frac{J_m\left(\frac{x_{mn}}{a}\rho\right)}{J_{m+1}(x_{mn})}. \tag{3.23-29}$$

Note that $J_{m+1}(x_{mn})$ isn't zero. With the eigenfunctions known, I can now use (Jackson 3.160) to find the expansion of the potential,

$$\Phi(\rho,\phi,z) = \frac{q}{4\pi\epsilon_0}G(\rho,\phi,z) = \frac{q}{\epsilon_0}\sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{\psi^*_{kmn}(\mathbf{x}')\psi_{kmn}(\mathbf{x}')}{\lambda_{kmn}}, \tag{3.23-30}$$

where I replaced the single sum with three (since I have three indices). Substituting (3.23-29) into (3.23-30) along with its eigenvalues (3.23-26), I find

$$\boxed{\Phi(\mathbf{x},\mathbf{x}') = \frac{2q}{\pi\epsilon_0 La^2}\sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}\sin\left(\frac{k\pi z}{L}\right)\sin\left(\frac{k\pi z'}{L}\right)J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J^2_{m+1}(x_{mn})}} \tag{3.23-31}$$

as the expansion of the potential, which is the same as the third representation (3.23-1).

Discussion: It is easy to relate the last expression to the first through the use of (Jackson 3.169):

$$\frac{\sinh(K_{lm}z_<)\sinh(K_{lm}(c-z_>))}{K_{lm}\sinh(K_{lm}c)} = \frac{2}{c}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)}{K^2_{lm}+\left(\frac{n\pi}{c}\right)^2} \tag{3.23-32}$$

This can be substituted into (3.23-31) under the assumption that $c = L$, $k \leftrightarrow n$ (index swap), and $K_{lm} \to x_{mn}/a$:

$$\begin{aligned}\Phi(\mathbf{x},\mathbf{x}') &= \frac{q}{\pi\epsilon_0 a^2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{J^2_{m+1}(x_{mn})}\times\frac{2}{L}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi z}{L}\right)\sin\left(\frac{k\pi z'}{L}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]}\\ &= \frac{q}{\pi\epsilon_0 a^2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{J^2_{m+1}(x_{mn})}\times\frac{a}{x_{mn}}\frac{\sinh\left(\frac{x_{mn}}{a}z_<\right)\sinh\left(\frac{x_{mn}}{a}(L-z_>)\right)}{\sinh\left(\frac{x_{mn}L}{a}\right)}\\ &= \frac{q}{\pi\epsilon_0 a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{x_{mn}J^2_{m+1}(x_{mn})\sinh\left(\frac{x_{mn}L}{a}\right)}\sinh\left[\frac{x_{mn}}{a}z_<\right]\sinh\left[\frac{x_{mn}}{a}(L-z_>)\right]\ \square\end{aligned} \tag{3.23-33}$$

which is indeed the first expression. This must be so, as the potential is unique. Note that the second and third expressions already have fairly similar forms, which if I equate the two I find the condition

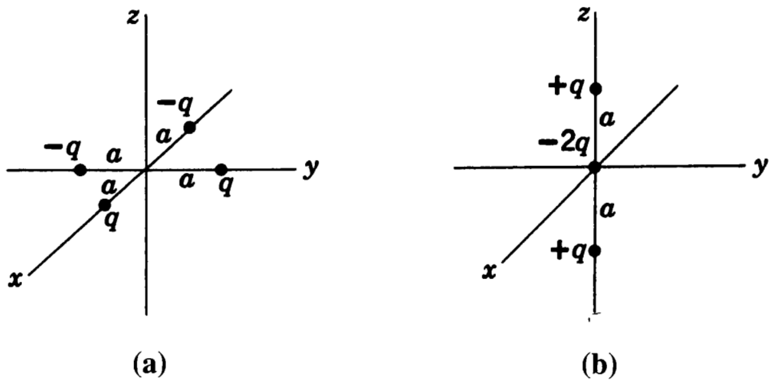
$$\frac{2}{a^2}\sum_{n=1}\frac{J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J^2_{m+1}(x_{mn})} = \frac{I_m\left(\frac{k\pi}{L}\rho_<\right)}{I_m\left(\frac{k\pi a}{L}\right)}\left[I_m\left(\frac{k\pi a}{L}\right)K_m\left(\frac{k\pi\rho_>}{L}\right)-K_m\left(\frac{k\pi a}{L}\right)I_m\left(\frac{k\pi\rho_>}{L}\right)\right] \tag{3.23-34}$$

It is not immediately apparent to me that this equation is true, but it must be, due to the uniqueness of the potential.

Chapter 4

Jackson (4.1)

Calculate the multipole moments q_{lm} of the charge distributions shown as parts a and b. Try to obtain the results for the nonvanishing moments valid for all l, but in each case find the first two sets of nonvanishing moments at the very least.



Solution (a): The formula for the multipole moments is

$$q_{lm} = \int_0^\infty dr' r'^2 \int_0^\pi d\theta' \sin\theta' \int_0^{2\pi} d\phi' \rho(r', \theta', \phi') r'^l Y_{lm}^*(\theta', \phi'). \tag{4.1-1}$$

It is simple to write the charge density as a sum of the four point particles,

$$\begin{aligned} \rho(r', \theta', \phi') = q\frac{\delta(r'-a)\delta(\theta'-\pi/2)\delta(\phi')}{a^2} - q\frac{\delta(r'-a)\delta(\theta'-\pi/2)\delta(\phi'-\pi)}{a^2} \\ + q\frac{\delta(r'-a)\delta(\theta'-\pi/2)\delta(\phi'-\pi/2)}{a^2} - q\frac{\delta(r'-a)\delta(\theta'-\pi/2)\delta(\phi'+\pi/2)}{a^2} \\ = \frac{q}{a^2}\delta(r'-a)\delta(\theta'-\pi/s)\left[\delta(\phi') - \delta(\phi'-\pi) + \delta(\phi'-\pi/2) - \delta(\phi'+\pi/2)\right]. \end{aligned} \tag{4.1-2}$$

where the denominator of each was simplified by using the radial and polar delta functions. Substituting this into (4.1-1) and acting with the radial and polar delta functions,

$$\begin{aligned} q_{lm} &= qa^l \int_0^{2\pi} d\phi' Y_{lm}^*(\frac{\pi}{2}, \phi') \left[\delta(\phi') - \delta(\phi'-\pi) + \delta(\phi'-\pi/2) - \delta(\phi'+\pi/2)\right] \\ &= qa^l \left[Y_{lm}^*(\frac{\pi}{2}, 0) - Y_{lm}^*(\frac{\pi}{2}, \pi) + Y_{lm}^*(\frac{\pi}{2}, \frac{\pi}{2}) - Y_{lm}^*(\frac{\pi}{2}, -\frac{\pi}{2})\right]. \end{aligned} \tag{4.1-3}$$

Using the fact that

$$Y_{lm}^*(\theta', \phi') = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} e^{-im\phi'} P_l^m(\cos\phi'), \tag{4.1-4}$$

I can simplify (4.1-3) to

$$\begin{aligned} q_{lm} &= qa^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(0) \left[1 - e^{-im\pi} + e^{-im\pi/2} - e^{im\pi/2}\right] \\ &= qa^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(0) \left[1 - (-1)^m - i^m(1 - (-1)^m)\right], \end{aligned} \tag{4.1-5}$$

which can be pulled together as

$$\boxed{q_{lm} = qa^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(0)(1 - (-1)^m)(1 - i^m)} \tag{4.1-6}$$

for distribution (a).

Solution (b): All that changes relative to (a) is the charge density, which now has the form

$$\begin{aligned}\rho(r',\theta',\phi') &= \frac{q}{2\pi}\frac{\delta(r'-a)\delta(\theta')}{a^2\sin\theta'} - \frac{2q}{4\pi}\frac{\delta(r')}{r'^2} + \frac{q}{2\pi}\frac{\delta(r'-a)\delta(\theta'-\pi)}{a^2\sin\theta'} \\ &= \frac{q}{2\pi a^2}\frac{\delta(r'-a)}{\sin\theta'}\left[\delta(\theta')+\delta(\theta'-\pi)\right] - \frac{q}{2\pi}\frac{\delta(r')}{r'^2},\end{aligned} \tag{4.1-7}$$

where due to azimuthal symmetry I divided each term by 2π. Note that each term has an apparent divergence, but in reality when substituted into (4.1-1) the divergences cancel out. Let's do just that,

$$\begin{aligned}q_{lm} = \frac{qa^l}{2\pi}\int_0^{\pi} d\theta'\left[\delta(\theta')+\delta(\theta'-\pi)\right]\int_0^{2\pi} d\phi' Y^*_{lm}(\theta',\phi') \\ - \frac{q}{2\pi}\int_0^{\infty} dr' r'^l\delta(r')\int_0^{\pi} d\theta'\sin\theta'\int_0^{2\pi} d\phi' Y^*_{lm}(\theta',\phi').\end{aligned} \tag{4.1-8}$$

Note that the integral $\int r'^l\delta(r') = \delta_{l,0}\delta_{m,0}$ (since m is bounded by l). Using this relation and simplifying the first term:

$$\begin{aligned}q_{lm} &= \frac{qa^l}{2\pi}\int_0^{2\pi} d\phi'\left[Y^*_{lm}(0,\phi')+Y^*_{lm}(\pi,\phi')\right] - \frac{q}{2\pi}\delta_{l,0}\delta_{m,0}\int_0^{\pi} d\theta'\sin\theta'\int_0^{2\pi} d\phi' Y^*_{00}(\theta',\phi') \\ &= -\frac{q}{\sqrt{\pi}}\delta_{l,0}\delta_{m,0} + \frac{qa^l}{2\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}\int_0^{2\pi} d\phi' e^{-im\phi'}\left[P_l^m(1)+P_l^m(-1)\right],\end{aligned} \tag{4.1-9}$$

Where I used the fact that $Y_{00} = 1/\sqrt{4\pi}$ is constant and the angular integrals in the second integral could be performed, as well as relation (4.1-4) for the first term. Parity for the associated Legendre polynomials tells me that $P_l^m(-1) = (-1)^{l+m}P_l^m(1)$, so I can simplify things,

$$\begin{aligned}q_{lm} &= -\frac{q}{\sqrt{\pi}}\delta_{l,0}\delta_{m,0} + \frac{qa^l}{2\pi}P_l^m(1)\left(1+(-1)^{l+m}\right)\sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}\int_0^{2\pi} d\phi' e^{-im\phi'} \\ &= -\frac{q}{\sqrt{\pi}}\delta_{l,0}\delta_{m,0} + \frac{qa^l}{2\pi}\sqrt{\frac{2l+1}{4\pi}}P_l^0(1)\left(1+(-1)^l\right)(2\pi\delta_{m,0}),\end{aligned} \tag{4.1-10}$$

$$\boxed{q_{lm} = -\frac{q}{\sqrt{\pi}}\delta_{l,0}\delta_{m,0} + qa^l\sqrt{\frac{2l+1}{4\pi}}\left(1+(-1)^l\right)\delta_{m,0}} \tag{4.1-11}$$

These are the multipole moments for distribution (b) valid for all l, m. Note that I took $P_l^0(1) = 1$.

(c): For the charge distribution of the second set b write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the $x-y$ plane as a function of distance from the origin for distances greater than a. (Divide out the asymptotic forms to see the large distance behavior clearly)

Solution: The multipole expansion in terms of the multipole moments has the form

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \\ &= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left[-\frac{q}{\sqrt{\pi}}\delta_{l,0}\delta_{m,0} + qa^l\sqrt{\frac{2l+1}{4\pi}}\left(1+(-1)^l\right)\delta_{m,0}\right]\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \\ &= -\frac{q}{\epsilon_0\sqrt{\pi}}\frac{Y_{00}(\theta,\phi)}{r} + \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sqrt{\frac{4\pi}{2l+1}}\frac{a^l}{r^{l+1}}(1+(-1)^l)Y_{l0}(\theta,\phi) \\ &= -\frac{q}{2\pi\epsilon_0 r} + \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{a^l}{r^{l+1}}(1+(-1)^l)P_l(\cos\theta),\end{aligned} \tag{4.1-12}$$

where I used the fact that $P_l^0(x) = P_l(x)$. This form can be simplified, as since $P_0(\cos\theta) = 1$, the first term in the sum cancels the bare term. This motivates a change of index $l = 2l$ (since $l = 1$ clearly vanishes),

$$\boxed{\Phi(\mathbf{x}) = \frac{2q}{4\pi\epsilon_0}\sum_{l=1}^{\infty}\frac{a^{2l}}{r^{2l+1}}P_{2l}(\cos\theta)} \tag{4.1-13}$$

This is the most compact form for the multipole expansion of the potential, which also avoids vanishing moments. For large distance behavior, only the $l = 1$ term is relevant:

$$\Phi(\mathbf{x}) \approx \frac{q}{4\pi\epsilon_0}\frac{a^2}{r^3}\left(3\cos^2\theta - 1\right) \xrightarrow{\theta=\pi/2} -\frac{q}{4\pi\epsilon_0}\frac{a^2}{r^3}, \qquad (r \gg a) \tag{4.1-14}$$

where in the last step I evaluated it in the xy-plane ($\theta = \pi/2$). **I am not going to plot this yet, but I will plot it alongside (d) for comparison (see Fig.1 below).**

(d): Calculate directly from Coulomb's law the exact potential for b in the $x - y$ plane. Plot it as a function of distance and compare with the result found in part c. (Divide out the asymptotic forms to see the large distance behavior clearly)

Solution: This is simple,

$$\begin{aligned}\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{||\mathbf{x} - a\hat{z}||} + \frac{1}{||\mathbf{x} + a\hat{z}||} - \frac{2}{||\mathbf{x}||}\right] &= \frac{q}{4\pi\epsilon_0}\left[\frac{2}{\sqrt{r^2 + a^2}} - \frac{2}{r}\right] \\ &= \frac{2q}{4\pi\epsilon_0}\frac{1}{r}\left[\frac{1}{\sqrt{1 + (a/r)^2}} - 1\right] \\ &\approx -\frac{q}{4\pi\epsilon_0}\frac{a^2}{r^3}\end{aligned} \tag{4.1-15}$$

which agrees nicely under the $r \gg a$ limit. I will now plot both (4.1-14) and the full expansion (4.1-15) together,

Figure 3: Exact potential $\Phi(r)$ (blue) and first order approximation (red) plotted for $a = 1$ in units such that $q/4\pi\epsilon_0 = 1$. At a distance of $r > 2a$ the curves are quite similar, but for $r \sim a$ the approximation fails.

Jackson (4.6)

A nucleus with quadrupole moment Q finds itself in a cylindrically symmetric electric field with a gradient $(\partial E_z/\partial z)_0$ along the z axis at the position of the nucleus.

(a) Show that the energy of quadrupole interaction is

$$W = -\frac{e}{4}Q\left(\frac{\partial E_z}{\partial z}\right)_0 \tag{4.6-1}$$

Solution: Jackson (4.24) tells me that the general formula for the quadrupole interaction energy is

$$W = -\frac{1}{6}\sum_{i,j} Q_{ij}\left(\frac{\partial E_j}{\partial x_i}\right)_0 . \tag{4.6-2}$$

On page 151, Jackson states that for a nucleus the non-zero quadrupole moments are $Q_{33} = eQ$, $Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$, where e is the protonic charge.[1] This means the sum (4.6-2) collapses to three terms,

$$W = -\frac{e}{6}Q\left(-\frac{1}{2}\left(\frac{\partial E_x}{\partial x}\right)_0 - \frac{1}{2}\left(\frac{\partial E_y}{\partial y}\right)_0 + \left(\frac{\partial E_z}{\partial z}\right)_0\right) \tag{4.6-3}$$

Since $\mathbf{E}$ is an external field, $\nabla \cdot \mathbf{E} = 0$, which gives the relation

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \iff \left(\frac{\partial E_x}{\partial x}\right)_0 + \left(\frac{\partial E_y}{\partial y}\right)_0 = -\left(\frac{\partial E_z}{\partial z}\right)_0 , \tag{4.6-4}$$

where I chose to evaluate it at the origin so that I may substitute it into (4.6-3) to give

$$W = -\frac{e}{6}Q\left(\frac{1}{2} + \frac{1}{2} + 1\right)\left(\frac{\partial E_z}{\partial z}\right)_0 \implies \boxed{W = -\frac{e}{4}Q\left(\frac{\partial E_z}{\partial z}\right)_0} \tag{4.6-5}$$

as desired.

(b) If it is known that $Q = 2 \times 10^{-28}$ m^2 and that W/h is 10 MHz, where h is Planck's constant, calculate $(\partial E_z/\partial z)_0$ in units of $e/4\pi\epsilon_0 a_0^3$, where $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 0.529 \times 10^{-10}$ m is the Bohr radius in hydrogen.

Solution: First, I will rewrite (4.6-5):

$$\left(\frac{\partial E_z}{\partial z}\right) = -\frac{4W}{eQ} = -4\left(\frac{W}{h}\right)\left(\frac{1}{Q}\right)\left(\frac{h}{e}\right), \tag{4.6-6}$$

Using $h = 2\pi\hbar$ and dividing by $e/4\pi\epsilon_0 a_0^3$,

$$\begin{aligned}\left(\frac{\partial E_z}{\partial z}\right)_0 \times \frac{1}{e/4\pi\epsilon_0 a_0^3} &= -8\pi\left(\frac{W}{h}\right)\left(\frac{1}{Q}\right)\left(\frac{4\pi\epsilon_0\hbar a_0^3}{e^2}\right) \\ &= -8\pi(137)\left(\frac{W}{h}\right)\left(\frac{1}{Q}\right)\left(\frac{a_0^3}{c}\right),\end{aligned} \tag{4.6-7}$$

where I recognized the fine structure constant $\alpha = e^2/4\pi\epsilon_0\hbar c \approx 1/137$. Plugging in the given numbers (with $c = 3 \times 10^8$ m/s) I find

$$\boxed{\left(\frac{\partial E_z}{\partial z}\right)_0 \times \frac{1}{e/4\pi\epsilon_0 a_0^3} \approx -0.085} \tag{4.6-8}$$

is $(\partial E_z/\partial z)_0$ in units of $e/4\pi\epsilon_0 a_0^3$, and is a dimensionless quantity.

[1] This is also the hint Woodard gave.

(c) Nuclear charge distributions can be approximated by a constant charge density throughout a spheroidal volume of semimajor axis a and semiminor axis b. Calculate the quadrupole moment of such a nucleus, assuming that the total charge is Ze. Given that Eu^{153} ($Z = 63$) has a quadrupole moment $Q = 2.5 \times 10^{-28}$ m^2 and a mean radius $R = (a+b)/2 = 7 \times 10^{-15}$ m, determine the fractional difference in radius, $(a-b)/R$.

Solution: Relative to (4.6-5), note that $Q = \frac{1}{e} Q_{33}$, so I simply need to compute Q_{33}. For this squashed sphere along the z-axis, the integration can be performed in cylindrical coordinates with the bounds

$$-a \le z \le a, \qquad 0 \le \rho \le b\sqrt{1 - \frac{z^2}{a^2}}, \qquad 0 \le \phi \le 2\pi, \tag{4.6-9}$$

such that, after assuming ρ_0 to be the density, Jackson (4.9) becomes

$$\begin{aligned} Q_{33} &= \rho_0 \int_0^{2\pi} d\phi \int_{-a}^{a} dz \int_0^{b\sqrt{1-z^2/a^2}} ds\; s \left(3z^2 - (s^2 + z^2)\right) \\ &= 2\pi\rho_0 \int_{-a}^{a} dz \left(b^2 z^2 \left(1 - \frac{z^2}{a^2}\right) - \frac{b^4}{4}\left(1 - \frac{z^2}{a^2}\right)^2\right) \\ &= 4\pi\rho_0 \int_0^a dz \left(b^2 z^2 - \frac{b^2}{a^2} z^4 - \frac{b^4}{4}\left(1 - \frac{2z^2}{a^2} + \frac{z^4}{a^4}\right)\right) \\ &= 4\pi\rho_0 \left(\frac{b^2 a^3}{3} - \frac{b^2 a^3}{5} - \frac{b^4}{4}\left(a - \frac{2}{3}a + \frac{1}{5}a\right)\right) \\ &= \frac{8\pi}{15}\rho_0 a b^2 \left(a^2 - b^2\right), \end{aligned} \tag{4.6-10}$$

where I noted $r^3 r^3 = z^2$ and $r^2 = s^2 + z^2$. As is standard, I seek to rewrite Q_{33} in terms of the total charge Ze, which can be found by direct integration,

$$\begin{aligned} Ze = \rho_0 \int_0^{2\pi} d\phi \int_{-a}^{a} dz \int_0^{b\sqrt{1-z^2/a^2}} ds\; s &= 2\pi\rho_0 b^2 \int_0^a dz \left(1 - \frac{z^2}{a^2}\right) \\ &= \frac{4\pi}{3}\rho_0 a b^2. \end{aligned} \tag{4.6-11}$$

This can be substituted into (4.6-10), from which $Q = Q_{33}/e$ follows:

$$Q_{33} = \frac{2Ze}{5}(a^2 - b^2) \implies \boxed{Q = \frac{2Z}{5}(a^2 - b^2)} \tag{4.6-12}$$

this is the quadrupole moment of the given nucleus assuming a total charge Ze. Using $Z = 63$, I will expand (4.6-12) and factor in such a way to make the fractional difference in radius apparent. Since the term in parentheses is a difference of two squares,

$$\begin{aligned} Q = \frac{2Z}{5} 2(a-b)\frac{(a+b)}{2} &= \frac{4Z}{5}(a-b)R, \\ \implies \frac{(a-b)}{R} &= \frac{5Q}{4ZR^2}. \end{aligned} \tag{4.6-13}$$

Using the given numbers, I find

$$\boxed{\frac{(a-b)}{R} \approx 0.101} \tag{4.6-14}$$

as the fractional difference in radius for Eu^{153}.

Jackson (4.8)

A very long, right circular, cylindrical shell of dielectric constant ϵ/ϵ_0 and inner and outer radii a and b, respectively, is placed in a previously uniform electric field $\mathbf{E}_0$ with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.

(a) Determine the potential and electric field in the three regions, neglecting end effects.

Solution: Assume that far away, $\mathbf{E} \to E_0\hat{x}$. Note that the cylinder is long (no z dependence), in polar coordinates then $\Phi = \Phi(s,\phi)$. There is no free charge in the problem, so $\nabla^2\Phi = 0$ and

$$\left[\partial_s^2 + \frac{1}{s}\partial_s + \frac{1}{s^2}\partial_\phi^2\right]\Phi(s,\phi) = 0. \tag{4.8-1}$$

So we have periodic boundary conditions in ϕ, and the homogenous cylindrical solution to Laplace's equation is s^m or $1/s^m$ so

$$\Phi(s,\phi) = \sum_{m=-\infty}^{\infty}\left(\alpha_m s^{|m|} + \frac{\beta_m}{s^{|m|}}\right)e^{im\phi} \tag{4.8-2}$$

is my guess. Note that the uniformity of the electric field at large distances implies the BC $\Phi \to -E_0 s\cos\phi$. Since only $\cos\phi$ is the source for the problem, the specialization of the radial potential to various regions is

$$\Phi(s) = \begin{cases} \alpha_m s^{|m|}, & 0 \le s \le a \\ \beta_m s^{|m|} + \frac{\gamma_m}{s^{|m|}}, & a \le s \le b \\ -\frac{E_0 s}{2}\delta_{|m|,1} + \frac{\eta_m}{s^{|m|}}, & s \ge b \end{cases} \tag{4.8-3}$$

and the $\delta_{|m|,1}$ effectively sets $|m| = 1$ for the entire problem. It follows, then, that

$$\begin{gathered}\Phi(s,\phi) = \begin{cases} \alpha s\cos\phi, & 0 \le s \le a \\ \left(\beta s + \frac{\gamma}{s}\right)\cos\phi, & a \le s \le b \\ \left(\frac{\eta}{s} - E_0 s\right)\cos\phi, & s \ge b \end{cases}, \\ \implies E_s = \begin{cases} -\alpha\cos\phi, & 0 \le s \le a \\ -\left(\beta - \frac{\gamma}{s^2}\right)\cos\phi, & a \le s \le b \\ \left(\frac{\eta}{s^2} + E_0\right)\cos\phi, & s \ge b \end{cases} \quad \& \quad E_\phi = \begin{cases} \alpha\sin\phi, & 0 \le s \le a \\ \left(\beta + \frac{\gamma}{s^2}\right)\sin\phi, & a \le s \le b \\ \left(\frac{\eta}{s^2} - E_0\right)\sin\phi, & s \ge b \end{cases}\end{gathered} \tag{4.8-4}$$

The BC's at $s = a$, $\Delta\Phi = 0$ and $\Delta D_s = 0$ imply

$$\alpha a = \beta a + \frac{\gamma}{a}, \qquad \epsilon_0\alpha = \epsilon\left(\beta - \frac{\gamma}{a^2}\right), \tag{4.8-5}$$

which can be manipulated to yield

$$\gamma = \left(\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\right)a^2\beta, \qquad \alpha = \left(\frac{2\epsilon}{\epsilon+\epsilon_0}\right)\beta. \tag{4.8-6}$$

The BC's at $s = b$ are slightly more involved,

$$\beta b + \frac{\gamma}{b} = \frac{\eta}{b} - E_0 b, \qquad \epsilon\left(-\beta + \frac{\gamma}{b^2}\right) = \epsilon_0\left(\frac{\eta}{b^2} + E_0\right), \tag{4.8-7}$$

which can be manipulated to eliminate η,

$$-2E_0 = \left(\frac{\epsilon+\epsilon_0}{\epsilon\epsilon_0}\right)\beta - \left(\frac{\epsilon-\epsilon_0}{\epsilon\epsilon_0}\right)\frac{\gamma}{b^2} \xrightarrow{(4.8-6)} \left[\left(\frac{\epsilon+\epsilon_0}{\epsilon\epsilon_0}\right) - \left(\frac{\epsilon-\epsilon_0}{\epsilon\epsilon_0}\right)\left(\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\right)\frac{a^2}{b^2}\right]\beta, \tag{4.8-8}$$

where I used (4.8-6) determined β implicitly. This relation can be rewritten in a simpler form, and all the constants follow using the above BC's:

$$\beta = -\frac{2\epsilon_0(\epsilon+\epsilon_0)E_0}{(\epsilon+\epsilon_0)^2 + (\epsilon-\epsilon_0)^2\frac{a^2}{b^2}}, \qquad \gamma = -\frac{2\epsilon_0(\epsilon-\epsilon_0)a^2E_0}{(\epsilon+\epsilon_0)^2 + (\epsilon-\epsilon_0)^2\frac{a^2}{b^2}}, \tag{4.8-9}$$

$$\alpha = -\frac{4\epsilon\epsilon_0 E_0}{(\epsilon+\epsilon_0)^2 + (\epsilon-\epsilon_0)^2\frac{a^2}{b^2}}, \qquad \eta = \frac{\left(\epsilon^2-\epsilon_0^2\right)\left(b^2-a^2\right)E_0}{(\epsilon+\epsilon_0)^2 + (\epsilon-\epsilon_0)^2\frac{a^2}{b^2}}, \tag{4.8-10}$$

where I suppressed the algebra for obvious reasons. Via (4.8-4), I find

$$\mathbf{E}_{0\le s\le a}(s,\phi) = -\alpha\hat{x} = \frac{4\epsilon\epsilon_0 E_0}{(\epsilon+\epsilon_0)^2 + (\epsilon-\epsilon_0)^2\frac{a^2}{b^2}}\hat{x} \tag{4.8-11}$$

$$\begin{aligned}\mathbf{E}_{a\le s\le b}(s,\phi) &= \left(-\beta + \frac{\gamma}{s^2}\right)\hat{x} + \frac{2\gamma}{s^2}\sin\phi\hat{\phi} \\ &= \left(\frac{2\epsilon_0(\epsilon+\epsilon_0)E_0}{(\epsilon+\epsilon_0)^2 + (\epsilon-\epsilon_0)^2\frac{a^2}{b^2}} - \frac{2\epsilon_0(\epsilon-\epsilon_0)E_0}{(\epsilon+\epsilon_0)^2 + (\epsilon-\epsilon_0)^2\frac{a^2}{b^2}}\frac{a^2}{s^2}\right)\hat{x} \\ &\qquad - \frac{4\epsilon_0(\epsilon-\epsilon_0)E_0}{(\epsilon+\epsilon_0)^2 + (\epsilon-\epsilon_0)^2\frac{a^2}{b^2}}\frac{a^2}{s^2}\sin\phi\hat{\phi}\end{aligned} \tag{4.8-12}$$

$$\begin{aligned}\mathbf{E}_{s\ge b}(s,\phi) &= E_0\hat{x} + \frac{\eta}{s^2}\hat{x} + \frac{2\eta}{s^2}\sin\phi\hat{\phi} \\ &= E_0\hat{x} + \frac{\left(\epsilon^2-\epsilon_0^2\right)\left(b^2-a^2\right)E_0}{(\epsilon+\epsilon_0)^2 + (\epsilon-\epsilon_0)^2\frac{a^2}{b^2}}\frac{1}{s^2}\hat{x} + \frac{2\left(\epsilon^2-\epsilon_0^2\right)\left(b^2-a^2\right)E_0}{(\epsilon+\epsilon_0)^2 + (\epsilon-\epsilon_0)^2\frac{a^2}{b^2}}\frac{1}{s^2}\sin\phi\hat{\phi},\end{aligned} \tag{4.8-13}$$

as the electric fields in each region. Note that my choice in mixing cartesian and polar coordinates is to manifest the idea that the electric field has deviations from the uniform direction $\hat{x}$ as curls through $\hat{\phi}$.

(b) Sketch the lines of force for a typical case of $b \approx 2a$.

Solution: The electric field inside the cylinder is still constant and uniform in the same direction as the previous field. Inside the bulk of the cylinder and outside the cylinder the field curls, yet for large s is approximately uniform in x. This looks something like the following:

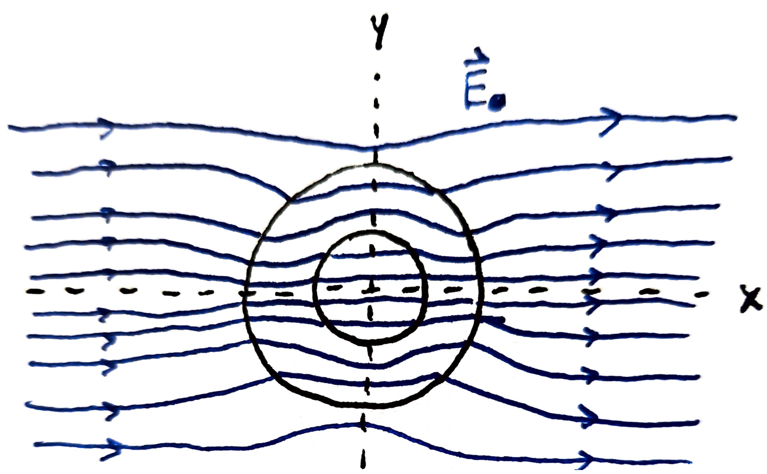


Figure 4: Electric field lines for a cylinder of dielectric constant ϵ/ϵ immersed in a previously uniform $\mathbf{E} = E_0\hat{x}$.

(c) Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

Solution: For a solid dielectric cylinder, then $a = 0$ and the fields become

$$\mathbf{E}_{0\le s\le b}(s,\phi) = \frac{2\epsilon_0 E_0}{\epsilon+\epsilon_0}\hat{x}, \qquad \mathbf{E}_{s\ge b}(s,\phi) = E_0\hat{x} + \frac{(\epsilon-\epsilon_0)\,E_0}{(\epsilon+\epsilon_0)}\frac{b^2}{s^2}\left[\hat{x} + 2\sin\phi\hat{\phi}\right], \tag{4.8-15}$$

where the electric field (4.8-11) vanishes. The electric field inside is uniform but diminished relative to the original field since $\epsilon > \epsilon_0$. For a cylindrical cavity in a uniform dielectric, $b \to \infty$ and

$$\mathbf{E}_{0\le s\le a}(s,\phi) = \frac{4\epsilon\epsilon_0 E_0}{(\epsilon+\epsilon_0)^2}\hat{x}, \qquad \mathbf{E}_{s\ge a}(s,\phi) = \frac{2\epsilon_0 E_0}{(\epsilon+\epsilon_0)}\hat{x} - \frac{2\epsilon_0(\epsilon-\epsilon_0)E_0}{(\epsilon+\epsilon_0)^2}\frac{a^2}{s^2}\left[\hat{x} + 2\sin\phi\hat{\phi}\right]. \tag{4.8-16}$$

This must align with (4.8-15) under $\epsilon \leftrightarrow \epsilon_0$ and $a \leftrightarrow b$. To see this, note that for the cavity the physical field measured at infinity is $E_0' = 2\epsilon_0 E_0/(\epsilon+\epsilon_0)$. Under this substitution, (4.8-16) becomes

$$\mathbf{E}_{0\le s\le a}(s,\phi) = \frac{2\epsilon E_0'}{\epsilon+\epsilon_0}\hat{x}, \qquad \mathbf{E}_{s\ge a}(s,\phi) = E_0'\hat{x} - \frac{(\epsilon-\epsilon_0)E_0'}{(\epsilon+\epsilon_0)}\frac{a^2}{s^2}\left[\hat{x} + 2\sin\phi\hat{\phi}\right], \tag{4.8-17}$$

which is precisely (4.8-15) under $\epsilon \leftrightarrow \epsilon_0$ and $a \leftrightarrow b$.

Jackson (4.9)

A point charge q is located in free space a distance d from the center of a dielectric sphere of radius a ($a < d$) and dielectric constant ϵ/ϵ_0.

(a) Find the potential at all points in space as an expansion in spherical harmonics.

Solution: I don't understand the need to expand in spherical harmonics when I can simply define my axes so that the point charge q is on the axis, such that azimuthal symmetry is apparent and I can expand in Legendre polynomials. The charge density induced in the dielectric by the point charge outside is only on the surface, and the volume bound charge is zero, so it suffices to use the homogeneous solution of the potential inside (and outside with an additional term). So inside, I find

$$\Phi_{\text{in}}(\mathbf{r}) = \frac{q}{4\pi\epsilon}\sum_{l=0}^{\infty}\alpha_l r^l P_l(\cos\theta), \tag{4.9-1}$$

where I neglected the $1/r^{l+1}$ contribution, and redefined my α_l by pulling a factor of $q/4\pi\epsilon$ outside. Outside of the sphere I will have the $1/r^{l+1}$ contribution plus the additional potential of a point charge,

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0||\mathbf{r}-\mathbf{r}'||} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta), \tag{4.9-2}$$

which can be combined the homogeneous solution outside:

$$\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left(\frac{\beta_l}{r^{l+1}} + \frac{r_<^l}{r_>^{l+1}}\right)P_l(\cos\theta). \tag{4.9-3}$$

(4.9-1) and (4.9-3) are the solutions for the potential inside and outside. With this surface bound charge we have the boundary conditions

$$\Phi_{\text{in}}(a) = \Phi_{\text{out}}(a), \qquad \epsilon\partial_r\Phi_{\text{in}}(\mathbf{r})\Big|_{r=a} = \epsilon_0\partial_r\Phi_{\text{out}}(\mathbf{r})\Big|_{r=a}. \tag{4.9-4}$$

Note that outside, $r > a$, and for the purposes of (4.9-4) $r < d$. Plugging things in, I find

$$\frac{1}{\epsilon}\alpha_l a^l = \frac{1}{\epsilon_0}\left(\frac{\beta_l}{a^{l+1}} + \frac{a^l}{d^{l+1}}\right), \qquad l\alpha_l a^{l-1} = -\frac{(l+1)\beta_l}{a^{l+2}} + \frac{la^{l-1}}{d^{l+1}}, \tag{4.9-5}$$

which can be algebraically solved to yield

$$\alpha_l = \frac{\epsilon}{d^{l+1}}\frac{2l+1}{\epsilon l + \epsilon_0(l+1)}, \qquad \beta_l = \frac{la^{2l+1}}{d^{l+1}}\frac{\epsilon_0 - \epsilon}{\epsilon l + \epsilon_0(l+1)}. \tag{4.9-6}$$

At this point I can plug in (4.9-6) into (4.9-1) and (4.9-3) to get the potential for all of space:

$$\boxed{\Phi_{\text{in}} = \frac{q}{4\pi\epsilon}\sum_{l=0}^{\infty}\frac{(2l+1)}{l+\frac{\epsilon_0}{\epsilon}(l+1)}\frac{r^l}{d^{l+1}}P_l(\cos\theta)} \tag{4.9-7}$$

$$\boxed{\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left(\frac{l\left(\frac{\epsilon_0}{\epsilon}-1\right)}{l+\frac{\epsilon_0}{\epsilon}(l+1)}\frac{a^{2l+1}}{d^{l+1}r^{l+1}} + \frac{r_<^l}{r_>^{l+1}}\right)P_l(\cos\theta)} \tag{4.9-8}$$

(b) Calculate the rectangular components of the electric field near the center of the sphere.

Solution: First, expand Φ_{in} to second-order in r (so the electric field is linear in r):

$$\Phi_{\text{in}}(\mathbf{r}\sim 0) = \frac{q}{4\pi\epsilon}\left[\frac{\epsilon}{\epsilon_0}\frac{1}{d} + \frac{3}{1+\frac{2\epsilon_0}{\epsilon}}\frac{z}{d^2} + \frac{5}{2+\frac{3\epsilon_0}{\epsilon}}\frac{r^2}{d^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\right], \tag{4.9-9}$$

which gives rise to

$$\begin{aligned}\mathbf{E}_{\text{in}}(\mathbf{r}\sim 0) = -\nabla\Phi_{\text{in}}(\mathbf{r}\sim 0) &= -\frac{q}{4\pi\epsilon}\nabla\left[\frac{\epsilon}{\epsilon_0}\frac{1}{d} + \frac{3}{1+\frac{2\epsilon_0}{\epsilon}}\frac{z}{d^2} + \frac{5}{2+\frac{3\epsilon_0}{\epsilon}}\frac{1}{d^3}\left(\frac{3}{2}z^2 - \frac{1}{2}r^2\right)\right], \\ &= -\frac{q}{4\pi\epsilon}\left[\frac{3}{1+\frac{2\epsilon_0}{\epsilon}}\frac{\hat{z}}{d^2} + \frac{5}{2+\frac{3\epsilon_0}{\epsilon}}\frac{1}{d^3}\left(3z\hat{z} - r\hat{r}\right)\right],\end{aligned} \tag{4.9-10}$$

or in rectangular components is

$$\boxed{\mathbf{E}_{\text{in}}(\mathbf{r}\sim 0) = -\frac{q}{4\pi\epsilon d^2}\left[\frac{3}{1+\frac{2\epsilon_0}{\epsilon}}\hat{z} - \frac{5}{2+\frac{3\epsilon_0}{\epsilon}}\frac{1}{d}(x\hat{x} + y\hat{y} - 2z\hat{z})\right]} \tag{4.9-11}$$

(c) Verify that, in the limit $\epsilon/\epsilon_0 \to \infty$, your result is the same as that for the conducting sphere.

Solution: For the potential Φ_{in}, only the $l = 0$ term survives under this limit,

$$\boxed{\Phi_{\text{in}}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 d}} \tag{4.9-12}$$

which is indeed the potential inside of a conducting sphere, which must be constant. Outside, I find (4.9-8) becomes

$$\Phi_{\text{out}}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) - \frac{q}{4\pi\epsilon_0}\sum_{l=1}^{\infty}\frac{a^{2l+1}}{(dr)^{l+1}}P_l(\cos\theta), \tag{4.9-13}$$

where I carefully noted that $l = 0$ drops one of the terms in (4.9-8), and needed to shift the index after performing the limit (as the explicit l dependence vanishes after the limit). Lowering the index back to $l = 0$ on the second sum and accounting for it,

$$\Phi_{\text{out}}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{a}{dr}\right) + \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) - \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{a^l}{(dr/a)^{l+1}}P_l(\cos\theta), \tag{4.9-14}$$

I can now use the formula for the expansion of $||\mathbf{r}-\mathbf{r}'||^{-1}$ in Legendre polynomials to simplify the two series. Importantly, I have $\cos\theta$ and not $\cos\gamma$, which is an artifact that $\theta' = 0$ (charge on the z-axis). So, when I write the $||\mathbf{r}-\mathbf{r}'||^{-1}$, I need to let $\mathbf{r}' = r'\hat{z}$, giving

$$\Phi_{\text{out}}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{a}{dr}\right) + \frac{q}{4\pi\epsilon_0}\frac{1}{||\mathbf{r} - d\hat{z}||} - \frac{q}{4\pi\epsilon_0}\frac{1}{||\frac{d}{a}\mathbf{r} - a\hat{z}||}, \tag{4.9-15}$$

where I noted that $a < dr/a$ outside the sphere. I could stop here, but to align with (2.8) of Jackson, define the quantity $\mathbf{y} = d\hat{z}$, such that

$$\boxed{\Phi_{\text{out}}(\mathbf{r}) = \frac{qa/y}{4\pi\epsilon_0}\frac{1}{||\mathbf{r}||} + \frac{q}{4\pi\epsilon_0}\frac{1}{||\mathbf{r} - \mathbf{y}||} - \frac{qa/y}{4\pi\epsilon_0}\frac{1}{||\mathbf{r} - \frac{a^2}{y^2}\mathbf{y}||}} \tag{4.9-16}$$

which is precisely (2.8) of Jackson, the potential outside of a conducting sphere for $Q = 0$ (no charge on surface). Note the appearance of the image charge $q' = -qa/y$ at distance a^2/y!

Jackson (4.13)

Two long, coaxial, cylindrical conducting surfaces of radii a and b are lowered vertically into a liquid dielectric. If the liquid rises an average height h between the electrodes when a potential difference V is established between them, show that the susceptibility of the liquid is

$$\chi_e = \frac{(b^2 - a^2)\rho g h \ln(b/a)}{\epsilon_0 V^2} \tag{4.13-1}$$

where ρ is the density of the liquid, g is the acceleration due to gravity, and the susceptibility of air is neglected.

Solution: It's evident that I will need the potential between the cylinders. The potential cannot depend on z because the cylinders are "long" and the potential shouldn't have any angular dependence by Gauss' Law (azimuthal symmetry). Laplace's equation is still satisfied inside the dielectric, which in cylindrical coordinates gives rise to the relation

$$0 = \frac{1}{s}\partial_s(s\partial_s\Phi) + \cancel{\frac{1}{s^2}\partial_\phi^2\Phi} + \cancel{\partial_z^2\Phi} \implies s\partial_s\Phi = \alpha, \tag{4.13-2}$$

where $\alpha \in \mathbb{R}$. Integrating this relation yields

$$\Phi(s) = \alpha \ln(s) + \beta, \tag{4.13-3}$$

for another arbitrary constant β. Put a potential of V on the inner surface and the potential of 0 on the outer surface,[2] which gives rise to boundary conditions $\Phi(a) = V$ and $\Phi(b) = 0$ such that

$$\begin{aligned} \Phi(b) = 0 &\implies \Phi(s) = \alpha \ln(s/b). \\ \therefore\ \Phi(a) = V &\implies \Phi(s) = V\frac{\ln(s/b)}{\ln(a/b)}, \end{aligned} \tag{4.13-4}$$

so the potential in between the surfaces is determined. The corresponding electric field and polarization is simply

$$\mathbf{E}(s) = -\frac{V}{\ln(a/b)}\frac{\hat{s}}{s} \implies \mathbf{P}(s) = -\frac{\epsilon_0\chi_e V}{\ln(a/b)}\frac{\hat{s}}{s} \tag{4.13-5}$$

from which follows the electrostatic energy density

$$u_E = -\frac{1}{2}\mathbf{P}\cdot\mathbf{E} = -\frac{\epsilon_0\chi_e V^2}{2\ln^2(a/b)s^2}. \tag{4.13-6}$$

The gravitational potential energy of the fluid is $u_G = \rho g V z$ at some height z. What is really occurring is that the liquid rises obliquely, steeper in regions of larger potential, so the height $z(s)$ is not constant. However, it is still determined by the energy minimization condition $u_E + u_G = 0$, which says that the energy induced to the system by lowering it into a dielectric is manifest in change in height of the liquid:

$$u_E = -u_G \implies z(s) = \frac{\epsilon_0\chi_e V^2}{2\rho g \ln^2(a/b)s^2}. \tag{4.13-7}$$

We are told that the average height is h, which can be readily computed from (4.13-7):

$$\begin{aligned} h = \frac{\int_a^b ds\ s z(s)}{\int_a^b dz\ s} &= \frac{2}{b^2 - a^2}\left(\frac{\epsilon_0\chi_e V^2}{2\rho g \ln^2(a/b)}\right)\int_a^b ds\ \frac{1}{s} \\ &= \left(\frac{\epsilon_0\chi_e V^2}{(b^2 - a^2)\rho g \ln(a/b)}\right), \end{aligned} \tag{4.13-8}$$

which can be rewritten to give the electric susceptibility

$$\boxed{\chi_e = \frac{(b^2 - a^2)\rho g h \ln(b/a)}{\epsilon_0 V^2}} \tag{4.13-9}$$

as desired.

[2] It doesn't matter which surface I denote at a higher potential or to which specific values I give them. What matters is the potential difference V.

Chapter 5

Jackson (5.1)

Starting with the differential expression

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} d\mathbf{r}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{5.1-1}$$

for the magnetic induction at the point P with coordinate $\mathbf{x}$ produced by an increment of current $Id\mathbf{l}'$ at $\mathbf{x}'$, show explicitly that for a closed loop carrying a current I the magnetic induction at P is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \nabla\Omega \tag{5.1-2}$$

where Ω is the solid angle subtended by the loop at the point P. This corresponds to a magnetic scalar potential, $\Phi_M = -\mu_0 I\Omega/4\pi$. The sign convention for the solid angle is that $\Omega > 0$ if the point P views the "inner" side of the surface spanning the loop, that is, if a unit normal $\hat{n}$ to the surface is defined by the direction of current flow via the right hand rule, $\Omega > 0$ if $\hat{n}$ points away from the point P, and negative otherwise. This is the same convention as in Section 1.6 for the electric dipole layer.

Solution: The solid angle has the form

$$\Omega(\mathbf{r}) = \int_S d\mathbf{a}' \cdot \nabla \left(\frac{1}{||\mathbf{r} - \mathbf{r}'||} \right), \tag{5.1-3}$$

where Ω is the solid angle viewed from the point $\mathbf{r}$, and $\mathbf{r}'$ is the vector to the loop which subtends the solid angle. Integrating (5.1-1) and performing some basic operations,

$$\begin{aligned} \mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\partial S} d\mathbf{r}' \times \nabla' \left(\frac{1}{||\mathbf{r} - \mathbf{r}'||} \right) &= -\frac{\mu_0 I}{4\pi} \int_{\partial S} d\mathbf{r}' \times \nabla \left(\frac{1}{||\mathbf{r} - \mathbf{r}'||} \right) \\ &= \frac{\mu_0 I}{4\pi} \int_{\partial S} \left[\nabla \times \left(\frac{d\mathbf{l}'}{||\mathbf{r} - \mathbf{r}'||} \right) - \frac{\cancel{(\nabla \times d\mathbf{r}'))}}{||\mathbf{r} - \mathbf{r}'||} \right] \\ &= \frac{\mu_0 I}{4\pi} \nabla \times \int_{\partial S} \frac{d\mathbf{r}'}{||\mathbf{r} - \mathbf{r}'||}. \end{aligned} \tag{5.1-4}$$

Note the following formula from the front of Jackson,[3]

$$\int_{\partial S} d\mathbf{r}' \psi = \int_S d\mathbf{a}' \times \nabla' \psi = -\int_S d\mathbf{a}' \times \nabla \psi, \tag{5.1-5}$$

giving a quantity which can be simplified via BAC-CAB:

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= -\frac{\mu_0 I}{4\pi} \int_S \left[d\mathbf{a}' \nabla^2 \left(\frac{1}{||\mathbf{r} - \mathbf{r}'||} \right) - \nabla \left(d\mathbf{a}' \cdot \nabla \left(\frac{1}{||\mathbf{r} - \mathbf{r}'||} \right) \right) \right] \\ &= -\frac{\mu_0 I}{4\pi} \int_S d\mathbf{a}' \delta^3(\mathbf{r} - \mathbf{r}') + \frac{\mu_0 I}{4\pi} \nabla \int_S d\mathbf{a}' \cdot \nabla \left(\frac{1}{||\mathbf{r} - \mathbf{r}'||} \right) \end{aligned} \tag{5.1-6}$$

The last integral is the solid angle, and if I assume $\mathbf{r} \notin S$, then the first integral vanishes, leaving

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4\pi} \nabla\Omega} \tag{5.1-7}$$

as desired.

[3] It's a rewrite of Stokes' Theorem.

Jackson (5.2)

A long, right cylindrical, ideal solenoid of arbitrary cross section is created by stacking a large number of identical current-carrying loops one above the other, with N coils per unit length and each loop carrying a current I.

(a): In the approximation that the solenoidal coil is an ideal current sheet and infinitely long, use Problem 5.1 to establish that at any point inside the coil the magnetic field is axial and equal to

$$H = NI,$$

and that $H = 0$ for any point outside the coil.

Solution: The auxiliary field of a single loop of wire has the form

$$\mathbf{H}_1(\mathbf{x}) = \frac{I}{4\pi}\nabla\Omega \tag{5.2-1}$$

such that the total auxiliary field of a set of loops with winding number per unit length N has the form

$$\mathbf{H}(\mathbf{x}) = N\int_{-\infty}^{\infty} dz'\ \mathbf{H}_1(\mathbf{x}) = \frac{NI}{4\pi}\int_{-\infty}^{\infty} dz'\ \nabla\Omega \tag{5.2-2}$$

Note that the solid angle itself is odd around the point $z = z'$. We have a gradient inside, and the radial and azimuthal terms do not alter the oddness of the z-dependence, and hence they vanish after integrating over all of z. This motivates the replacement $\nabla = \hat{z}\frac{\partial}{\partial z}$

$$\mathbf{H}(\mathbf{x}) = \frac{NI}{4\pi}\int_{-\infty}^{\infty} dz'\frac{\partial\Omega}{\partial z}\hat{z} \tag{5.2-3}$$

Since $\frac{\partial}{\partial z}\Omega(z'-z) = -\frac{\partial}{\partial z'}\Omega(z'-z)$,

$$\begin{aligned}\mathbf{H}(\mathbf{x}) &= -\frac{NI}{4\pi}\int_{-\infty}^{\infty} dz'\frac{\partial\Omega}{\partial z'}\hat{z}\\ &= -\frac{NI}{4\pi}\left(\int_{-\infty}^{z-\epsilon} dz'\frac{\partial\Omega}{\partial z'}\hat{z} + \int_{z+\epsilon}^{\infty} dz'\frac{\partial\Omega}{\partial z'}\hat{z}\right)\\ &= -\frac{NI}{4\pi}\left(\Omega\Big|_{z'=\infty} - \Omega\Big|_{z'=z+\epsilon} + \Omega\Big|_{z'=z-\epsilon} - \Omega\Big|_{z'=-\infty}\right)\\ &= \frac{NI}{4\pi}\left(\Omega\Big|_{z'=z+\epsilon} - \Omega\Big|_{z'=z-\epsilon}\right)\end{aligned} \tag{5.2-4}$$

where I broke the integration point around the point of issue $z = z'$ then used FTC1. With the way the sign of the solid angle is defined, since I am above and below the surface these terms add. The solid angle inside a loop would have the form of 2π, and outside is zero, so (5.2-4) becomes

$$\boxed{\mathbf{H}(\mathbf{x}) = \begin{cases} NI\hat{z}, & \text{inside}\\ 0, & \text{outside}\end{cases}} \tag{5.2-5}$$

(b): For a realistic solenoid of circular cross section of radius a ($Na \gg 1$), but still infinite in length, show that the "smoothed" magnetic field just outside the solenoid (averaged axially over several turns) is not zero, but is the same in magnitude and direction as that of a single wire on the axis carrying a current I, even if $Na \to \infty$. Compare fields inside and out.

Solution: The parameterization of the solenoid is

$$\mathbf{l}(\lambda) = R\cos(\lambda - \phi)\hat{s} + R\sin(\lambda - \phi)\hat{\phi} + \frac{\lambda}{2\pi N}\hat{z} \tag{5.2-6}$$

for some parameter $\lambda \in \{-\infty, \infty\}$. I will perform a change of variables on the Biot-Savart Law:

$$\mathbf{H}(\mathbf{x}) = \frac{I}{4\pi}\oint_C \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{l})}{||\mathbf{r} - \mathbf{l}||^3} = \frac{I}{4\pi}\int_{-\infty}^{\infty} d\lambda \frac{\frac{d\mathbf{l}}{d\lambda} \times (\mathbf{r} - \mathbf{l})}{||\mathbf{r} - \mathbf{l}||^3}, \tag{5.2-7}$$

where since

$$\begin{aligned} \frac{d\mathbf{l}}{d\lambda} &= -R\sin(\lambda - \phi)\hat{s} + R\cos(\lambda - \phi)\hat{\phi} + \frac{1}{2\pi N}\hat{z} \\ \mathbf{r} - \mathbf{l} &= (s - R\cos(\lambda - \phi))\,\hat{s} - R\sin(\lambda - \phi)\hat{\phi} + \left(z - \frac{\lambda}{2\pi N}\right)\hat{z} \end{aligned} \tag{5.2-8}$$

the cross product becomes

$$\begin{aligned} \frac{d\mathbf{l}}{d\lambda} \times (\mathbf{r} - \mathbf{l}) &= R^2\sin^2(\lambda - \phi)\hat{z} + R\sin(\lambda - \phi)\left(z - \frac{\lambda}{2\pi N}\right)\hat{\phi} \\ &\quad - R\cos(\lambda - \phi)(s - R\cos(\lambda - \phi))\hat{z} + R\cos(\lambda - \phi)\left(z - \frac{\lambda}{2\pi N}\right)\hat{s} \\ &\quad + \frac{1}{2\pi N}\left(s - R\cos(\lambda - \phi)\right)\hat{\phi} + \frac{1}{2\pi N}R\sin(\lambda - \phi)\hat{s} \\ &= \hat{s}\frac{R}{2\pi N}\left[\sin(\lambda - \phi) + 2\pi N z\cos(\lambda - \phi) - \lambda\cos(\lambda - \phi)\right] \\ &\quad + \hat{\phi}\left[Rz\sin(\lambda - \phi) - \frac{R}{2\pi N}\lambda\sin(\lambda - \phi) + \frac{1}{2\pi N}s - \frac{R}{2\pi N}\cos(\lambda - \phi)\right] \\ &\quad + \hat{z}R\left[R - s\cos(\lambda - \phi)\right], \end{aligned} \tag{5.2-9}$$

which I will break into components:

$$\begin{aligned} \langle H\rangle_s(\mathbf{x}) = \frac{I}{4\pi}\frac{R}{4\pi^2 N}\int_{-\infty}^{\infty} d\lambda \int_0^{2\pi} d\phi \Big[&\frac{\sin(\phi)}{\left(s^2 + R^2 - 2Rs\cos(\phi) + \left(z - \frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} \\ + \frac{2\pi N z\cos(\phi)}{\left(s^2 + R^2 - 2Rs\cos(\phi) + \left(z - \frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} &- \frac{\lambda\cos(\phi)}{\left(s^2 + R^2 - 2Rs\cos(\phi) + \left(z - \frac{\lambda}{2\pi N}\right)^2\right)^{3/2}}\Big] \end{aligned} \tag{5.2-10}$$

$$\begin{aligned} \langle H\rangle_\phi(\mathbf{x}) = \frac{I}{8\pi^2}\int_{-\infty}^{\infty} d\lambda \int_0^{2\pi} d\phi \Big[&\frac{Rz\sin(\phi)}{\left(s^2 + R^2 - 2Rs\cos(\phi) + \left(z - \frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} \\ - \frac{R}{2\pi N} &\frac{\lambda\sin(\phi)}{\left(s^2 + R^2 - 2Rs\cos(\phi) + \left(z - \frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} \\ + \frac{s}{2\pi N} &\frac{1}{\left(s^2 + R^2 - 2Rs\cos(\phi) + \left(z - \frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} \\ - \frac{R}{2\pi N} &\frac{\cos(\phi)}{\left(s^2 + R^2 - 2Rs\cos(\phi) + \left(z - \frac{\lambda}{2\pi N}\right)^2\right)^{3/2}}\Big] \end{aligned} \tag{5.2-11}$$

$$\langle H\rangle_z(\mathbf{x}) = \frac{IR}{8\pi^2}\int_{-\infty}^{\infty} d\lambda \int_0^{2\pi} d\phi \Big[\frac{R}{\left(s^2+R^2-2Rs\cos(\phi)+\left(z-\frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} - \frac{s\cos(\phi)}{\left(s^2+R^2-2Rs\cos(\phi)+\left(z-\frac{\lambda}{2\pi N}\right)^2\right)^{3/2}}\Big] \tag{5.2-12}$$

I will now evaluate all the various types of integrals separately. It is simple to show that

$$\int_0^{2\pi} d\phi \, \frac{\sin(\phi)}{\left(s^2+R^2-2Rs\cos(\phi)+\left(z-\frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} = 0 \tag{5.2-13}$$

after the u-substitution of the denominator. In addition,

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\lambda \int_0^{2\pi} d\phi \frac{1}{\left(s^2+R^2-2Rs\cos(\phi)+\left(z-\frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} \\
&= \int_{-\infty}^{\infty} d\lambda \int_0^{2\pi} d\phi \frac{1}{\left(s^2+R^2-2Rs\cos(\phi)\right)^{3/2}} \frac{1}{\left(1+\frac{\left(z-\frac{\lambda}{2\pi N}\right)^2}{s^2+R^2-2Rs\cos(\phi)}\right)^{3/2}} \\
&= 2\pi N \int_0^{2\pi} \frac{d\phi}{\left(s^2+R^2-2Rs\cos(\phi)\right)} \int_{-\pi/2}^{\pi/2} d\alpha \cos\alpha \\
&= 4\pi N \int_0^{2\pi} \frac{d\phi}{\left(s^2+R^2-2Rs\cos(\phi)\right)} \\
&= \frac{8\pi^2 N}{|s^2-R^2|},
\end{aligned} \tag{5.2-14}$$

as well as

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\lambda \int_0^{2\pi} d\phi \frac{\cos\phi}{\left(s^2+R^2-2Rs\cos(\phi)+\left(z-\frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} \\
&= -\frac{1}{2Rs}\int_{-\infty}^{\infty} d\lambda \int_0^{2\pi} d\phi \frac{s^2+R^2-2Rs\cos\phi}{\left(s^2+R^2-2Rs\cos(\phi)+\left(z-\frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} \\
&+ \frac{s^2+R^2}{2Rs}\int_{-\infty}^{\infty} d\lambda \int_0^{2\pi} d\phi \frac{1}{\left(s^2+R^2-2Rs\cos(\phi)+\left(z-\frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} \\
&= -\frac{\pi N}{Rs}\int_{-\pi/2}^{\pi/2} d\alpha \cos\alpha \int_0^{2\pi} d\phi + \frac{s^2+R^2}{2Rs}\frac{8\pi^2 N}{|s^2-R^2|} \\
&= -\frac{4\pi^2 N}{Rs} + \frac{s^2+R^2}{2Rs}\frac{8\pi^2 N}{|s^2-R^2|}.
\end{aligned} \tag{5.2-15}$$

Note also that

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\lambda \int_0^{2\pi} d\phi \frac{\lambda\cos\phi}{\left(s^2+R^2-2Rs\cos(\phi)+\left(z-\frac{\lambda}{2\pi N}\right)^2\right)^{3/2}} \\
&= 2\pi N \int_0^{2\pi} d\phi \frac{\cos\phi}{(s^2+R^2-2Rs\cos\phi)} \int_{-\pi/2}^{\pi/2} d\alpha\, 2\pi N \left(z+\sqrt{s^2+R^2-2Rs\cos\phi}\tan\alpha\right)\cos\alpha \\
&= 8\pi^2N^2z \int_0^{2\pi} d\phi \frac{\cos\phi}{(s^2+R^2-2Rs\cos\phi)} \\
&= -\frac{8\pi^2N^2z}{2Rs} \int_0^{2\pi} d\phi \frac{s^2+R^2-2Rs\cos\phi-(s^2+R^2)}{(s^2+R^2-2Rs\cos\phi)} \\
&= -\frac{8\pi^2N^2z}{2Rs} \int_0^{2\pi} d\phi \left(1-\frac{s^2+R^2}{(s^2+R^2-2Rs\cos\phi)}\right) \\
&= -\frac{8\pi^2N^2z}{2Rs}\left(2\pi-\frac{2\pi(s^2+R^2)}{|s^2-R^2|}\right).
\end{aligned} \tag{5.2-16}$$

Therefore, the s, ϕ, z components of the magnetic field can be evaluated:

$$\begin{gathered}
\langle H\rangle_s(\mathbf{x}) = \frac{I}{4\pi}\frac{R}{4\pi^2N}\left(\frac{s^2+R^2}{2Rs}\frac{16\pi^3N^2z}{|s^2-R^2|}-\frac{8\pi^3N^2z}{Rs}+\frac{16\pi^3N^2z}{2Rs}-\frac{s^2+R^2}{2Rs}\frac{16\pi^3N^2z}{|s^2-R^2|}\right) \\
\therefore\ \boxed{\langle H\rangle_s(\mathbf{x})=0}
\end{gathered} \tag{5.2-17}$$

$$\begin{gathered}
\langle H\rangle_\phi(\mathbf{x}) = \frac{I}{8\pi^2}\left[\frac{s}{2\pi N}\left(\frac{8\pi^2N}{|s^2-R^2|}\right)-\frac{R}{2\pi N}\left(\frac{8\pi^2N}{|s^2-R^2|}\frac{s^2+R^2}{2Rs}-\frac{4\pi^2N}{Rs}\right)\right] \\
\therefore\ \boxed{\langle H\rangle_\phi(\mathbf{x})=\frac{I}{4\pi s}\left[1+\frac{s^2-R^2}{|s^2-R^2|}\right]}
\end{gathered} \tag{5.2-18}$$

$$\begin{gathered}
\langle H\rangle_z(\mathbf{x}) = \frac{IR}{8\pi^2}\left[\frac{8\pi^2NR}{|s^2-R^2|}+\frac{4\pi^2N}{R}-\frac{s^2+R^2}{2R}\frac{8\pi^2N}{|s^2-R^2|}\right] \\
\therefore\ \boxed{\langle H\rangle_z(\mathbf{x})=\frac{IN}{2}\left[1-\frac{s^2-R^2}{|s^2-R^2|}\right]}
\end{gathered} \tag{5.2-19}$$

So the axially averaged radial component is zero for all of space. Now, for inside then $s^2-R^2<0$ and

$$\langle H\rangle_{\phi,\text{in}}(\mathbf{x}) = \frac{I}{4\pi s}\left[1-\frac{R^2-s^2}{R^2-s^2}\right]=0, \qquad \langle H\rangle_{z,\text{in}}(\mathbf{x}) = \frac{IN}{2}\left[1+\frac{R^2-s^2}{R^2-s^2}\right]=NI \tag{5.2-20}$$

and for outside $s^2-R^2>0$,

$$\langle H\rangle_{\phi,\text{out}}(\mathbf{x}) = \frac{I}{4\pi s}\left[1+\frac{s^2-R^2}{s^2-R^2}\right]=\frac{I}{2\pi s}, \qquad \langle H\rangle_{z,\text{out}}(\mathbf{x}) = \frac{IN}{2}\left[1-\frac{s^2-R^2}{s^2-R^2}\right]=0 \tag{5.2-21}$$

Jackson (5.3)

A right-circular solenoid of a finite length L and radius a has N turns per unit length and carries a current I. Show that the magnetic induction on the cylinder axis in the limit $NL \to \infty$ is

$$B_z = \frac{\mu_0 N I}{2} \left(\cos\theta_1 + \cos\theta_2\right), \tag{5.3-1}$$

where the angles are defined in the figure

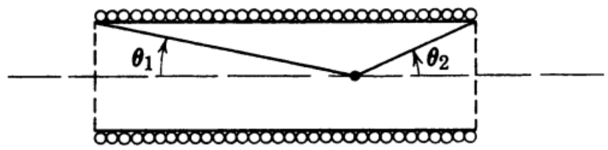


Figure 5: Jackson (5.3)

Solution: I will first find the magnetic field of a single loop in the xy-plane of radius r as a function of z, then integrate it over the length of the solenoid. Here, $\mathbf{r}' = a\cos\phi'\hat{x} + a\sin\phi'\hat{y}$, so since $\mathbf{r} = s\hat{s} + z\hat{z}$ (no angular), then $\mathbf{r} - \mathbf{r}' = (s\cos\phi - a\cos\phi')\hat{x} + (s\sin\phi - a\sin\phi')\hat{y} - z\hat{z}$ and

$$||\mathbf{r} - \mathbf{r}'|| = \sqrt{s^2 - 2sa\cos(\phi - \phi') + a^2 + z^2} \tag{5.3-2}$$

Note additionally that, only taking the z-component,

$$\begin{aligned}
[d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')]_z &= [(-a\sin\phi'\hat{x} + a\cos\phi'\hat{y}) \times ((s\cos\phi - a\cos\phi')\hat{x} + (s\sin\phi - a\sin\phi')\hat{y} - z\hat{z})]_z \\
&= [-a\sin\phi'(s\sin\phi - a\sin\phi') - a\cos\phi'(s\cos\phi - a\cos\phi')]\,\hat{z} \\
&= \left[a^2 - as\cos(\phi - \phi')\right]\hat{z},
\end{aligned} \tag{5.3-3}$$

so Biot-Savart's law (on cylinder axis $s = 0$) reads as

$$\begin{aligned}
B_z^{(1)} = \frac{\mu_0 I}{4\pi} \lim_{s\to 0} \int_0^{2\pi} d\phi' \frac{a^2 - as\cos(\phi - \phi')}{\left(s^2 - 2as\cos(\phi - \phi') + a^2 + z^2\right)^{3/2}} &= \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\alpha \frac{a^2}{\left(a^2 + z^2\right)^{3/2}} \\
&= \frac{\mu_0 I}{2} \frac{a^2}{\left(a^2 + z^2\right)^{3/2}},
\end{aligned} \tag{5.3-4}$$

for the z-component magnetic field field of a loop of radius a in the xy-plane evaluated on the axis. The field of the solenoid is obtained by integrating loops at z' each evaluated at z,

$$\begin{aligned}
B_z = N \int_0^L dz' B_z^{(1)} &= \frac{\mu_0 N I}{2} \int_0^L dz' \frac{a^2}{\left(a^2 + (z - z')^2\right)^{3/2}} \\
&= \frac{\mu_0 N I}{2} \int_{\tan^{-1}(-z/a)}^{\tan^{-1}((L-z)/a)} d\alpha \cos\alpha \\
&= \frac{\mu_0 N I}{2} \left(\sin\tan^{-1}\left(\frac{L - z}{a}\right) - \sin\tan^{-1}\left(-\frac{z}{a}\right)\right) \\
&= \frac{\mu_0 N I}{2} \left(\frac{L - z}{\sqrt{(L - z)^2 + a^2}} + \frac{z}{\sqrt{L^2 + a^2}}\right) \implies \boxed{B_z = \frac{\mu_0 N I}{2} \left(\cos\theta_1 + \cos\theta_2\right)}
\end{aligned} \tag{5.3-5}$$

which agrees with the problem statement.

Jackson (5.4)

A magnetic induction $\mathbf{B}$ in a current-free region in a uniform medium is cylindrically symmetric with components $B_z(\rho, z)$ and $B_\rho(\rho, z)$ and with a known $B_z(0, z)$ on the axis of symmetry. The magnitude of the axial field varies slowly in z.

(a) Show that near the axis the axial and radial components of magnetic induction are approximately

$$\begin{aligned} B_z(\rho, z) &\approx B_z(0, z) - \left(\frac{\rho^2}{4}\right)\left[\frac{\partial^2 B_z(0, z)}{\partial z^2}\right] + \cdots \\ B_\rho(\rho, z) &\approx -\left(\frac{\rho}{2}\right)\left[\frac{\partial B_z(0, z)}{\partial z}\right] + \left(\frac{\rho^3}{16}\right)\left[\frac{\partial^3 B_z(0, z)}{\partial z^3}\right] + \cdots \end{aligned} \tag{5.4-1}$$

Solution: No ϕ dependence on the magnetic induction as a result of azimuthal symmetry. Since there is no current,

$$\nabla \times \mathbf{B} = 0 \implies \partial_z B_s = \partial_s B_z. \tag{5.4-2}$$

The idea is to expand both B_z in a Taylor series centered around $s = 0$,[4]

$$B_z(s, z) = \sum_{n \geq 0} \frac{s^n}{n!} \frac{\partial^n B_z(0, z)}{\partial s^n}, \tag{5.4-3}$$

then note that since $\nabla^2 \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla \times (\nabla \times \mathbf{B}) = 0$ in the absence of current, then

$$0 = \nabla^2 B_z(s, z) = \frac{1}{s}\partial_s(s \partial_s B_z) + \partial_z^2 B_z, \tag{5.4-4}$$

where since from (5.4-3) I find

$$\begin{aligned} \frac{1}{s}\partial_s(s\partial_s B_z) = \sum_{n\geq 1} \frac{n^2 s^{n-2}}{n!} \frac{\partial^n B_z(0, z)}{\partial s^n} &= \frac{1}{s}\frac{\partial}{\partial s} B_z(0, z) + \sum_{n \geq 2} \frac{n^2 s^{n-2}}{n!} \frac{\partial^n B_z(0, z)}{\partial s^n} \\ &= \frac{1}{s}\frac{\partial}{\partial s} B_z(0, z) + \sum_{n \geq 0} \frac{(n+2)s^n}{(n+1)n!} \frac{\partial^{n+2} B_z(0, z)}{\partial s^{n+2}}, \end{aligned} \tag{5.4-5}$$

then (5.4-4) reads as

$$0 = \frac{1}{s}\frac{\partial}{\partial s} B_z(0, z) + \sum_{n \geq 0} \frac{s^n}{n!}\left[\frac{\partial^2}{\partial z^2}\frac{\partial^n B_z(0, z)}{\partial s^n} + \frac{(n+2)}{(n+1)}\frac{\partial^{n+2} B_z(0, z)}{\partial s^{n+2}}\right]. \tag{5.4-6}$$

This relation must hold $\forall s$, and since neither terms in the sum have the same s for any n, the two vanish separately,

$$\frac{\partial B_z(0, z)}{\partial s} = 0, \qquad \frac{\partial^{n+2} B_z(0, z)}{\partial s^{n+2}} = -\frac{(n+1)}{(n+2)}\frac{\partial^2}{\partial z^2}\frac{\partial^n B_z(0, z)}{\partial s^n}, \tag{5.4-7}$$

the second of which being a recursion relation. To learn from this recursion relation, I will continue it for n iterations:

$$\begin{aligned} \frac{\partial^{n+2} B_z(0, z)}{\partial s^{n+2}} &= \left[-\frac{(n+1)}{(n+2)}\frac{\partial^2}{\partial z^2}\right]\left[-\frac{(n-1)}{(n)}\frac{\partial^2}{\partial z^2}\right]\left[-\frac{(n-3)}{(n-2)}\frac{\partial^2}{\partial z^2}\right]\frac{\partial^{n-4} B_z(0, z)}{\partial s^{n-4}} \\ &= (-1)^{n/2+1}\frac{(n+1)!!}{(n+2)!!}\frac{\partial^{n+2} B_z(0, z)}{\partial z^{n+2}} \qquad (n \text{ even}) \end{aligned} \tag{5.4-8}$$

This is valid for even n, let $n + 2 \to 2n$ such that

$$\frac{\partial^{2n} B_z(0, z)}{\partial s^{2n}} = (-1)^n \frac{(2n-1)!!}{(2n)!!}\frac{\partial^{2n} B_z(0, z)}{\partial z^{2n}}, \qquad (\forall n) \tag{5.4-9}$$

where since

$$\frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{((2n)!!)^2} = \frac{(2n)!}{2^{2n}(n!)^2}, \tag{5.4-10}$$

[4] The B_s series is redundant since $B_s(0, z)$ is zero (who says what direction in s it points?)

then (5.4-3) says that

$$
\begin{aligned}
B_z(s,z) = \sum_{n\geq 0} \frac{s^n}{n!}\frac{\partial^n B_z(0,z)}{\partial s^n} &= \cancel{\sum_{n\geq 0} \frac{s^{2n+1}}{(2n+1)!}\frac{\partial^{2n_1} B_z(0,z)}{\partial s^{2n+1}}} + \sum_{n\geq 0} \frac{s^{2n}}{(2n)!}\frac{\partial^{2n} B_z(0,z)}{\partial s^{2n}} \\
&= \sum_{n\geq 0} \frac{s^{2n}}{(2n)!}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\frac{\partial^{2n} B_z(0,z)}{\partial z^{2n}},
\end{aligned} \tag{5.4-11}
$$

$$
\boxed{B_z(s,z) = \sum_{n\geq 0} \frac{(-1)^n s^{2n}}{2^{2n}(n!)^2}\frac{\partial^{2n} B_z(0,z)}{\partial z^{2n}}} \tag{5.4-12}
$$

meaning the z-component of the magnetic field for all of space is determined uniquely by the z component of the magnetic field on the symmetry axis. It is simple to show that this quantity, when expanded to $n = 1$, gives precisely (5.4-1). In determining the series expansion for $B_s(s,z)$, I only need to use the curl condition (5.4-2):

$$
\begin{aligned}
\frac{\partial B_s(s,z)}{\partial z} = \frac{\partial B_z(s,z)}{\partial s} &= \sum_{n\geq 1} \frac{(-1)^n n s^{2n-1}}{2^{2n-1}(n!)^2}\frac{\partial^{2n} B_z(0,z)}{\partial z^{2n}} \\
&= \frac{\partial}{\partial z}\sum_{n\geq 1} \frac{(-1)^n s^{2n-1}}{2^{2n-1}n!(n-1)!}\frac{\partial^{2n-1} B_z(0,z)}{\partial z^{2n-1}} \\
\therefore\ B_s(s,z) &= \sum_{n\geq 1} \frac{(-1)^n s^{2n-1}}{2^{2n-1}n!(n-1)!}\frac{\partial^{2n-1} B_z(0,z)}{\partial z^{2n-1}},
\end{aligned} \tag{5.4-13}
$$

which can be simplified using an index shift,

$$
\boxed{B_s(s,z) = \sum_{n\geq 0} \frac{(-1)^{n+1} s^{2n+1}}{2^{2n+1}n!(n+1)!}\frac{\partial^{2n+1} B_z(0,z)}{\partial z^{2n+1}}} \tag{5.4-14}
$$

meaning the s-component of the magnetic field for all space is determined uniquely by the z-component of the magnetic field on the symmetry axis. Since the magnetic field has no angular dependence, (5.4-12) and (5.4-14) both imply that the knowledge of the z-magnetic field on the symmetry axis determines the magnetic field for all of space! It is simple to show that when expanded to $n = 1$, (5.4-14) gives (5.4-1).

Jackson (5.8)

A localized cylindrically symmetric current distribution is such that the current flows only in the azimuthal direction; the current density is a function only of r, θ (or ρ and z): $\mathbf{J} = J(r,\theta)\hat{\phi}$. The distribution is "hollow" in the sense that there is a current-free region near the origin, as well as outside.

(a) Show that the magnetic field can be derived from the azimuthal component of the vector potential, with a multipole expansion

$$A_\phi(r,\theta) = \begin{cases} -\frac{\mu_0}{4\pi}\sum_L m_L r^L P_L^1(\cos\theta), & \text{interior} \\ -\frac{\mu_0}{4\pi}\sum_L \mu_L r^{-L-1} P_L^1(\cos\theta), & \text{outside} \end{cases} \tag{5.8-1}$$

(b) Show that the internal and external multipole moments are

$$m_L = -\frac{1}{L(L+1)}\int d^3r\, r^{-L-1}P_L^1(\cos\theta)J(r,\theta), \qquad \mu_L = -\frac{1}{L(L+1)}\int d^3r\, r^L P_L^1(\cos\theta)J(r,\theta). \tag{5.8-2}$$

Solution: I don't know why Jackson calls the index L and not l but whatever. Also, part (b) is a one-liner from part (a) so this is in general a very weirdly worded problem. Nonetheless, the vector potential has the form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{J(r',\theta')\hat{\phi}'}{||\mathbf{r}-\mathbf{r}'||} \tag{5.8-3}$$

for the given current distribution. Note that $\hat{\phi}'$ depends on ϕ',

$$\hat{\phi}' = -\sin\phi'\hat{x} + \cos\phi'\hat{y} = \sin(\phi-\phi')\hat{s} + \cos(\phi-\phi')\hat{\phi}, \tag{5.8-4}$$

and must be integrated over. Expanding the distance factor in Legendre polynomials,

$$\frac{1}{||\mathbf{r}-\mathbf{r}'||} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)P_l^m(\cos\theta')e^{im(\phi-\phi')}, \tag{5.8-5}$$

gives

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_m^l(\cos\theta)\int_0^{2\pi}d\phi' e^{im(\phi-\phi')}\hat{\phi}'\int_{-1}^{1}d(\cos\theta')\int_0^\infty dr' r'^2 \times \frac{r_<^l}{r_>^{l+1}}P_l^m(\cos\theta')J(r',\theta'). \tag{5.8-6}$$

The azimuthal integration follows,

$$\begin{aligned} \int_0^{2\pi}d\phi' e^{im(\phi-\phi')}\hat{\phi}' &= \int_0^{2\pi}d\phi'\,[\cos[m(\phi-\phi')] + i\sin[m(\phi-\phi')]]\left[\sin(\phi-\phi')\hat{s} + \cos(\phi-\phi')\hat{\phi}\right] \\ &= \delta_{|m|,1}\int_0^{2\pi}d\phi'\left[\cos^2(\phi-\phi')\hat{\phi} + \text{sign}(m)i\sin^2(\phi-\phi')\hat{s}\right] \\ &= \pi\delta_{|m|,1}\left[\hat{\phi} + \text{sign}(m)i\hat{s}\right], \end{aligned} \tag{5.8-7}$$

so after recalling that $P_l^{-m} = (-1)^m(l-m)!/(l+m)! \times P_l^m$, (5.8-6) becomes

$$\begin{aligned} \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}&\left[\frac{(l-1)!}{(l+1)!}P_l^1(\cos\theta)\pi[\hat{\phi}+i\hat{s}] + \frac{(l+1)!}{(l-1)!}\left((-1)\frac{(l-1)!}{(l+1)!}\right)^2 P_l^1(\cos\theta)\pi[\hat{\phi}-i\hat{s}]\right] \\ &\times\int_{-1}^{1}d(\cos\theta')\int_0^\infty dr' r'^2\times\frac{r_<^l}{r_>^{l+1}}P_l^1(\cos\theta')J(r',\theta') \\ = \frac{\mu_0\hat{\phi}}{4\pi}\sum_{l=0}^{\infty}&\frac{2\pi}{l(l+1)}P_l^1(\cos\theta)\int_{-1}^{1}d(\cos\theta')\int_0^\infty dr' r'^2\times\frac{r_<^l}{r_>^{l+1}}P_l^1(\cos\theta')J(r',\theta'). \end{aligned} \tag{5.8-8}$$

Denoting $2\pi = \int_0^{2\pi} d\phi'$, I can re-combine the remaining integrals as a single volume integral, bringing constants inside to give

$$A_\phi(\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} P_l^1(\cos\theta) \int d^3r' \, \frac{r_<^l}{r_>^{l+1}} \frac{P_l^1(\cos\theta') J(r',\theta')}{l(l+1)}. \tag{5.8-9}$$

Inside, $r_< = r$ and $r_> = r'$ so

$$\begin{aligned} A_{\phi,\mathrm{in}}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} r^l P_l^1(\cos\theta) \int d^3r' r'^{-l-1} \frac{P_l^1(\cos\theta') J(r',\theta')}{l(l+1)} \\ &\equiv -\frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} m_l r^l P_l^1(\cos\theta), \end{aligned} \tag{5.8-10}$$

which agrees with the problem statement through the definition of the interior multipole moment

$$\boxed{m_l = -\frac{1}{l(l+1)} \int d^3r' r'^{-l-1} P_l^1(\cos\theta') J(r',\theta')} \tag{5.8-11}$$

as desired. Outside, $r_< = r'$ and $r_> = r$ so

$$\begin{aligned} A_{\phi,\mathrm{out}}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} r^{-l-1} P_l^1(\cos\theta) \int d^3r' r'^l \frac{P_l^1(\cos\theta') J(r',\theta')}{l(l+1)} \\ &\equiv -\frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \mu_l r^{-l-1} P_l^1(\cos\theta), \end{aligned} \tag{5.8-12}$$

which agrees with the problem statement through the definition of the exterior multipole moment

$$\boxed{\mu_l = -\frac{1}{l(l+1)} \int d^3r' r'^l P_l^1(\cos\theta') J(r',\theta')} \tag{5.8-13}$$

as desired.

Jackson (5.11)

A circular loop of wire carrying a current I is located with its center at the origin of coordinates and the normal to its plane having spherical angles θ_0, ϕ_0. There is an applied magnetic field, $B_x = B_0(1+\beta y)$ and $B_y = B_0(1+\beta x)$.

(a): Calculate the force acting on the loop without making any approximations. Compare your result with the approximate result (5.69). Comment.

Solution: The given parameterization $\mathbf{l}(\lambda)$ can be written in the x, y, z basis:

$$\begin{aligned}\mathbf{l}(\lambda) &= R\cos\lambda\hat{\theta}_0 + R\sin\lambda\hat{\phi}_0 \\ &= R\left(\cos\lambda\cos\theta_0\cos\phi_0 - \sin\lambda\sin\phi_0\right)\hat{x} + R\left(\cos\lambda\cos\theta_0\sin\phi_0 + \sin\lambda\cos\phi_0\right)\hat{y} \\ &\qquad - R\cos\lambda\sin\theta_0\hat{z}\end{aligned} \tag{5.11-1}$$

$$\begin{aligned}\implies \frac{d\mathbf{l}}{d\lambda} &= -R\left(\sin\lambda\cos\theta_0\cos\phi_0 + \cos\lambda\sin\phi_0\right)\hat{x} - R\left(\sin\lambda\cos\theta_0\sin\phi_0 - \cos\lambda\cos\phi_0\right)\hat{y} \\ &\qquad + R\sin\lambda\sin\theta_0\hat{z}\end{aligned} \tag{5.11-2}$$

The Lorentz force law in the case of a linear current density has the form $d\mathbf{F} = I d\mathbf{l}\times\mathbf{B} = I\frac{d\mathbf{l}}{d\lambda}\times\mathbf{B}\,d\lambda$. Since $\mathbf{B} = B_0(1+\beta y)\hat{x} + B_0(1+\beta x)\hat{y}$, this curl evaluates to

$$\begin{aligned}\frac{d\mathbf{l}}{d\lambda}\times\mathbf{B} &= -B_0R(1+\beta x)\sin\lambda\sin\theta_0\hat{x} + B_0R(1+\beta y)\sin\lambda\sin\theta_0\hat{y} \\ &\quad - B_0R(1+\beta x)\left[\sin\lambda\cos\theta_0\cos\phi_0 + \cos\lambda\sin\phi_0\right]\hat{z} \\ &\quad + B_0R(1+\beta y)\left[\sin\lambda\cos\theta_0\sin\phi_0 - \cos\lambda\cos\phi_0\right]\hat{z}\end{aligned} \tag{5.11-3}$$

Since x, y are the coefficients of the corresponding basis vectors in (5.11-1), this substitution becomes messy. Thankfully, I know that I will eventually be integrating $\lambda \in \{0, 2\pi\}$, which means that only the $\sin^2\lambda$ and $\cos^2\lambda$ terms remain (and are π). In making this substitution to (5.11-3) I will only keep those factors:

$$\begin{aligned}\frac{d\mathbf{l}}{d\lambda}\times\mathbf{B} &= R^2B_0\beta\sin^2\lambda\sin\theta_0\sin\phi_0\hat{x} + R^2B_0\beta\sin^2\lambda\sin\theta_0\cos\phi_0\hat{y} \\ &\quad - 2R^2B_0\beta\left(\cos^2\lambda - \sin^2\lambda\right)\cos\theta_0\sin\phi_0\cos\phi_0\hat{z}\end{aligned} \tag{5.11-4}$$

Integration over λ implies the replacement $\sin^2\lambda \to \cos^2\lambda \to \pi$, which cancels the z-dependence in (5.11-4) multiplying this result by the current I, I find

$$\boxed{\mathbf{F} = \pi I R^2 B_0\beta\sin\theta_0(\sin\phi_0\hat{x} + \cos\phi_0\hat{y})} \tag{5.11-5}$$

The approximate formula for the force is $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$, and the magnetic moment is obvious:

$$\begin{aligned}\mathbf{m} &= \int I d\mathbf{a} = \pi I R^2\hat{n} \\ &= \pi I R^2\sin\theta_0\cos\phi_0\hat{x} + \pi I R^2\sin\theta_0\sin\phi_0\hat{y} + \pi I R^2\cos\theta_0\hat{z}.\end{aligned} \tag{5.11-6}$$

where I used the representation of the unit radial vector in spherical coordinates. Therefore,

$$\begin{aligned}\mathbf{m}\cdot\mathbf{B} &= \pi I R^2B_0\sin\theta_0\cos\phi_0\,(1+\beta y) + \pi I R^2B_0\sin\theta_0\sin\phi_0\,(1+\beta x) \\ \implies \nabla(\mathbf{m}\cdot\mathbf{B}) &= \pi I R^2B_0\beta\sin\theta_0(\sin\phi_0\hat{x} + \cos\phi_0\hat{y}),\end{aligned} \tag{5.11-7}$$

which is precisely the force (5.11-5). So in this case, $\nabla(\mathbf{m}\cdot\mathbf{B})$ is not an approximation. This gives a hint at the non-existence of higher order terms.

(b): Calculate the torque in lowest order. Can you deduce anything about the higher order contributions? Do they vanish for the circular loop? What about for other shapes?

Solution: To lowest order the torque is that of a magnetic dipole, $\tau = \mathbf{m} \times \mathbf{B}$, which using (5.11-6) this is easy to evaluate,

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B} = \pi I R^2 B_0 \left[\sin\theta_0 \cos\phi_0 \hat{x} + \sin\theta_0 \sin\phi_0 \hat{y} + \cos\theta_0 \hat{z}\right] \times [\hat{x} + \hat{y}], \tag{5.11-8}$$

$$\boxed{\boldsymbol{\tau} = \pi I R^2 B_0 \left[\cos\theta_0(-\hat{x} + \hat{y}) - \sin\theta_0(\sin\phi_0 - \cos\phi_0)\hat{z}\right]} \tag{5.11-9}$$

Is this what I get when I use the general formula for the torque? Well,

$$\begin{aligned} \boldsymbol{\tau} = I \int_0^{2\pi} d\lambda \, \mathbf{l} \times \left[\frac{d\mathbf{l}}{d\lambda} \times \mathbf{B}\right] &= I \int_0^{2\pi} d\lambda \left[\frac{d\mathbf{l}}{d\lambda}(\mathbf{l} \cdot \mathbf{B}) - \mathbf{B}\left(\mathbf{l} \cdot \frac{d\mathbf{l}}{d\lambda}\right)\right] \\ &= I \int_0^{2\pi} d\lambda \frac{d\mathbf{l}}{d\lambda}(\mathbf{l} \cdot \mathbf{B}) \end{aligned} \tag{5.11-10}$$

where since

$$\begin{aligned} \mathbf{l} \cdot \mathbf{B} = \Big[& R\left(\cos\lambda \cos\theta_0 \cos\phi_0 - \sin\lambda \sin\phi_0\right)\hat{x} + R\left(\cos\lambda \cos\theta_0 \sin\phi_0 + \sin\lambda \cos\phi_0\right)\hat{y} \\ & - R\cos\lambda \sin\theta_0 \hat{z}\Big] \cdot \left[B_0(1 + \beta y)\hat{x} + B_0(1 + \beta x)\hat{y}\right] \\ = {} & RB_0(1 + \beta y)\left(\cos\lambda \cos\theta_0 \cos\phi_0 - \sin\lambda \sin\phi_0\right) + RB_0(1 + \beta x)\left(\cos\lambda \cos\theta_0 \sin\phi_0 + \sin\lambda \cos\phi_0\right), \end{aligned} \tag{5.11-11}$$

a further multiplication of $\frac{d\mathbf{l}}{d\lambda}$ results in terms with 2 or 3 trig functions. Subsequent integration (5.11-10) kills all the terms with 3 trig functions, meaning I can safely take $\beta = 0$ beforehand in (5.11-11) and then multiply by $\frac{d\mathbf{l}}{d\lambda}$:

$$\begin{aligned} \frac{d\mathbf{l}}{d\lambda}(\mathbf{l} \cdot \mathbf{B}) = {} & -R^2 B_0 \left[(\cos\lambda \cos\theta_0 \cos\phi_0 - \sin\lambda \sin\phi_0) + (\cos\lambda \cos\theta_0 \sin\phi_0 + \sin\lambda \cos\phi_0)\right] \\ & \times \Big[(\sin\lambda \cos\theta_0 \cos\phi_0 + \cos\lambda \sin\phi_0)\hat{x} + (\sin\lambda \cos\theta_0 \sin\phi_0 - \cos\lambda \cos\phi_0)\hat{y} \\ & - \sin\lambda \sin\theta_0 \hat{z}\Big]. \end{aligned} \tag{5.11-12}$$

Keeping only the $\cos^2$ and $\sin^2$ terms results in

$$\begin{aligned} \frac{d\mathbf{l}}{d\lambda}(\mathbf{l} \cdot \mathbf{B}) = {} & -R^2 B_0 \hat{x}\Big[\left(\cos^2\lambda - \sin^2\lambda\right)\cos\theta_0 \sin\phi_0 \cos\phi_0 + \cos\theta_0\left(\cos^2\lambda \sin^2\phi_0 + \sin^2\lambda \cos^2\phi_0\right) \Big] \\ & - R^2 B_0 \hat{y}\Big[- \cos\theta_0\left(\cos^2\lambda \cos^2\phi_0 + \sin^2\lambda \sin^2\phi_0\right) - \left(\cos^2\lambda - \sin^2\lambda\right)\cos\theta_0 \sin\phi_0 \cos\phi_0\Big] \\ & - R^2 B_0 \hat{z} \sin^2\lambda \sin\theta_0 \Big[\sin\phi_0 - \cos\phi_0\Big], \end{aligned} \tag{5.11-13}$$

which when substituted into the formula for the torque (5.11-10) amounts to the replacement of $\cos^2 \to \sin^2 \to \pi$, giving the torque

$$\boldsymbol{\tau} = -\pi I R^2 B_0 \cos\theta_0 \hat{x} + \pi I R^2 B_0 \cos\theta_0 \hat{y} - \pi I R^2 B_0 \hat{z} \sin\theta_0 \Big[\sin\phi_0 - \cos\phi_0\Big], \tag{5.11-14}$$

$$\boxed{\boldsymbol{\tau} = \pi I R^2 B_0 \left[\cos\theta_0(-\hat{x} + \hat{y}) - \sin\theta_0(\sin\phi_0 - \cos\phi_0)\hat{z}\right]} \tag{5.11-15}$$

which is precisely the same as the lowest order (dipole) contribution (5.11-9), meaning the dipole approximation is exact and all higher order contributions must vanish. For other shapes of arbitrary symmetry, the higher order terms need not vanish necessarily.

Jackson (5.14)

A long, hollow, right circular cylinder of inner(outer) radius $a(b)$, and of relative permeability μ_r, is placed in a region of initially uniform magnetic-flux density $\mathbf{B}_0$ at right angles to the field. Find the flux density at all points in space, and sketch the logarithm of the ratio of the magnitudes $\mathbf{B}$ on the cylinder axis to $\mathbf{B}_0$ as a function of $\log_{10}(\mu)$, for $a^2/b^2 = 0.5, 0.1$. Neglect end effects.

Solution: The absence of free current means the curl of the magnetic field vanishes, $\nabla \times \mathbf{H} = 0$, which allows me to define a scalar magnetic potential $\mathbf{H} = -\nabla\Phi$. By non magnetic monopoles[5], it has an equation of motion $\nabla^2\Phi = 0$, just like in electrostatics. Due to cylindrical symmetry, I can take the potential to have the form

$$\Phi(s,\phi) = \sum_{m=-\infty}^{\infty} e^{im\phi}\Phi_m(s) \tag{5.14-1}$$

such that Laplace's equation reads

$$\left[\partial_s^2 + \frac{1}{s}\partial_s - \frac{m^2}{s^2}\right]\Phi_m(s) = 0 \implies \Phi_m(s) = \alpha_m s^{|m|} + \frac{\beta_m}{s^{|m|}} \tag{5.14-2}$$

In order to match the asymptotic behavior of $\mathbf{B} \to B_0\hat{x}$ I need to add the potential $-\frac{B_0}{\mu_0}s\cos\phi$, which to match the index structure (5.14-2) implies the presence of a delta function $\delta_{|m|,1}$ (only $\cos\theta$). Since this is the only source in the problem, all other m's vanish, and I can safely restrict myself to the $|m| = 1$ case (dropping all m labels).

$$\Phi(s,\phi) = \begin{cases} \alpha s\cos\phi, & (0 \le s \le a) \\ \left(\beta s + \frac{\gamma}{s}\right)\cos\phi, & (a \le s \le b) \\ \left(\frac{\eta}{s} - \frac{B_0 s}{\mu_0}\right)\cos\phi, & (s \ge b) \end{cases} \tag{5.14-3}$$

$$H_s(s,\phi) = \begin{cases} -\alpha\cos\phi, & (0 \le s \le a) \\ \left(-\beta + \frac{\gamma}{s^2}\right)\cos\phi, & (a \le s \le b) \\ \left(\frac{\eta}{s^2} + \frac{B_0}{\mu_0}\right)\cos\phi, & (s \ge b) \end{cases}, \qquad H_\phi(s,\phi) = \begin{cases} \alpha\sin\phi, & (0 \le s \le a) \\ \left(\beta + \frac{\gamma}{s^2}\right)\sin\phi, & (a \le s \le b) \\ \left(\frac{\eta}{s^2} - \frac{B_0}{\mu_0}\right)\sin\phi, & (s \ge b) \end{cases} \tag{5.14-4}$$

When appealing to the 2 boundary conditions at each of the 2 surfaces,

$$\Phi_{\text{in}} = \Phi_{\text{out}}, \qquad \mu_{\text{in}}H_{s,\text{in}} = \mu_{\text{out}}H_{s,\text{out}}, \tag{5.14-5}$$

I find 4 equations in 4 unknowns,

$$\begin{aligned} -a^2\alpha + a^2\beta + \gamma &= 0 \\ b^2\beta + \gamma - \eta &= -\frac{B_0 b^2}{\mu_0} \\ a^2\alpha - \frac{\mu a^2}{\mu_0}\beta + \frac{\mu}{\mu_0}\gamma &= 0 \\ \frac{\mu b^2}{\mu_0}\beta - \frac{\mu}{\mu_0}\gamma + \eta &= -\frac{B_0 b^2}{\mu_0} \end{aligned} \tag{5.14-6}$$

I then added equations (1) and (3) and then equations (2) and (4) to get two new equations that together determined γ.

[5] Possibly unjustified assumption.

I found the coefficients as

$$\begin{aligned}
\alpha &= -\frac{4\mu B_0 b^2}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2} \\
\beta &= -\frac{2B_0 b^2(\mu_0+\mu)}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2} \\
\gamma &= \frac{2B_0 a^2 b^2(\mu_0-\mu)}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2} \\
\eta &= \frac{B_0 b^2}{\mu_0}\frac{(b^2-a^2)(\mu^2-\mu_0^2)}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}
\end{aligned} \tag{5.14-7}$$

With these coefficients I have fully determined the radial and azimuthal components of the magnetic field H, and since $B_i = \mu_i H_i$, I have thus found the magnetic flux density. Substituting things in,

$$\mathbf{H}_{s\le a} = \frac{4\mu B_0 b^2}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}\left[\cos\phi\hat{s} - \sin\phi\hat{\phi}\right], \tag{5.14-8}$$

$$\therefore \boxed{\mathbf{B}_{s\le a} = \frac{4\mu_0\mu B_0 b^2}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}\hat{x}} \tag{5.14-9}$$

so the interior region has a uniform magnetic field in the $+\hat{x}$ direction.

$$\begin{aligned}
\mathbf{H}_{a\le s\le b} = &\frac{2B_0 b^2(\mu_0+\mu)}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}\hat{x} + \frac{2B_0 a^2 b^2(\mu_0-\mu)}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}\frac{1}{s^2}\hat{x} \\
&+ \frac{4B_0 a^2 b^2(\mu_0-\mu)}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}\frac{1}{s^2}\sin\phi\hat{\phi}
\end{aligned} \tag{5.14-10}$$

$$\therefore \boxed{\mathbf{B}_{a\le s\le b} = \frac{2\mu B_0 b^2}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}\left[\left(\mu_0+\mu+\frac{a^2}{s^2}(\mu_0-\mu)\right)\hat{x} + 2a^2(\mu_0-\mu)\frac{1}{s^2}\sin\phi\hat{\phi}\right]} \tag{5.14-11}$$

So within the dielectric, the magnetic field has both an x dependence and a ϕ dependence, where the ϕ dependence induces a curling of the field relative to the asymptotic $\hat{x}$ behavior. I know it's funny to mix cartesian and cylindrical coordinates, but I like the fact that it allows me to visual the magnetic field as a deviation from straight line behavior. We have one more to go,

$$\begin{aligned}
\mathbf{H}_{s\ge b} = &\frac{B_0}{\mu_0}\hat{x} + \frac{B_0 b^2}{\mu_0}\frac{(b^2-a^2)(\mu^2-\mu_0^2)}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}\frac{1}{s^2}\hat{x} \\
&+ \frac{2B_0 b^2}{\mu_0}\frac{(b^2-a^2)(\mu^2-\mu_0^2)}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}\frac{1}{s^2}\sin\phi\hat{\phi}
\end{aligned} \tag{5.14-12}$$

$$\therefore \boxed{\mathbf{B}_{s\ge b} = B_0\left[\left(1+\frac{(b^2-a^2)(\mu^2-\mu_0^2)}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}\frac{b^2}{s^2}\right)\hat{x} + \frac{2(b^2-a^2)(\mu^2-\mu_0^2)}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2}\frac{b^2}{s^2}\sin\phi\hat{\phi}\right]} \tag{5.14-13}$$

Again, I find the same curling behavior relative to the asymptotic $\hat{x}$ behavior, which is manifest through the limit $s\to\infty$, in which I find $\mathbf{B}_\infty \to B_0\hat{x}$ (as I should). Equations (2.9), (5.14-11) and (5.14-13) represent the magnetic flux density in all of space.

The ratio of the magnetic flux density on the axis to the initial value B_0 is straightforward from (5.14-9):

$$\begin{aligned}
\frac{B}{B_0} &= \frac{4\mu_0\mu b^2}{b^2(\mu_0+\mu)^2 - a^2(\mu_0-\mu)^2} \\
&= \frac{4\mu_r}{(1+\mu_r)^2 - \frac{a^2}{b^2}(1-\mu_r)^2}
\end{aligned} \tag{5.14-14}$$

where I took the given relative permeability $\mu_r = \mu/\mu_0$. Jackson wishes for me to make a log-log plot (base-10) of this function for the choices $a^2/b^2 = 0.1, 0.5$, which I will show below:

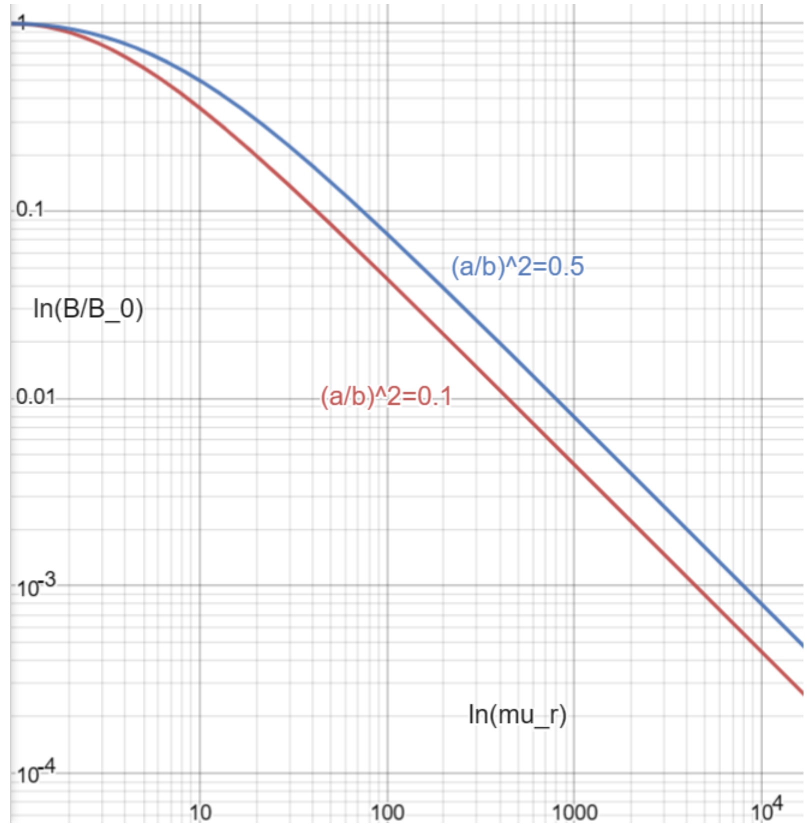


Figure 6: Log-Log plot of the ratio B/B_0 as a function of relative permeability. $(a/b)^2 = 0.5$ (blue) and $(a/b)^2 = 0.1$ (blue). At relative permeability of $\mu_r \sim 1000$ the interior field has diminished by $\sim 1/100$ of its initial value. This negative linear behavior (with slope ≈ -1) in the log-log plot implies that $B/B_0 \to \mu_r^{-1}$ asymptotically

Jackson (5.19)

A magnetically "hard" material is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization M_0 uniform throughout its volume and parallel to its axis.

(a) Determine the magnetic field $\mathbf{H}$ and magnetic induction $\mathbf{B}$ at all points on the axis of the cylinder, both inside and outside.

Solution: For hard magnets in the absence of current we have $\nabla^2\Phi_m = -\rho_m = -\nabla \cdot \mathbf{M}$, where ρ_m is the magnetic charge density. This equation maps to

$$\Phi_m = -\frac{1}{4\pi}\int d^3r' \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{||\mathbf{r}-\mathbf{r}'||}, \tag{5.19-1}$$

where $\mathbf{H} = -\nabla\Phi_m$ determines the magnetic field. The magnetization can be represented as $\mathbf{M} = M_0\hat{z}\Theta(a-s)\Theta(z)\Theta(L-z)$, such that

$$\nabla \cdot \mathbf{M} = M_0\Theta(a-s)\left[\delta(z) - \delta(L-z)\right] \tag{5.19-2}$$

and (5.19-1) becomes

$$\begin{aligned}\Phi_m &= -\frac{M_0}{4\pi}\int_0^{2\pi} d\phi' \int_{-\infty}^{\infty} dz' \int_0^{\infty} ds' s' \frac{\Theta(a-s)\left[\delta(z')-\delta(L-z')\right]}{\sqrt{s^2 - 2ss'\cos(\phi-\phi') + s'^2 + (z-z')^2}} \\ &= -\frac{M_0}{4\pi}\int_0^{2\pi} d\phi' \int_0^{a} ds' s' \left[\frac{1}{\sqrt{s^2+z^2-2ss'\cos(\phi-\phi')+s'^2}} - \frac{1}{\sqrt{s^2+(z-L)^2-2ss'\cos(\phi-\phi')+s'^2}}\right].\end{aligned} \tag{5.19-3}$$

Jackson requests that this be evaluated on the axis of the cylinder, so $s = 0$ and

$$\begin{aligned}\Phi_m(0,z) &= -\frac{M_0}{2}\int_0^a ds'\left[\frac{s'}{\sqrt{z^2+s'^2}} - \frac{s'}{\sqrt{(z-L)^2+s'^2}}\right] \\ &= -\frac{M_0}{2}\left[\int_{z^2}^{z^2+a^2}\frac{du}{2}u^{-1/2} - \int_{(z-L)^2}^{(z-L)^2+a^2}\frac{du}{2}u^{-1/2}\right] \\ &= -\frac{M_0}{2}\left[|z-L| - |z| + \sqrt{a^2+z^2} - \sqrt{a^2+(L-z)^2}\right].\end{aligned} \tag{5.19-4}$$

The magnetic field follows naturally,

$$\mathbf{H}(0,z) = -\hat{z}\frac{\partial}{\partial z}\Phi_m(0,z) = \frac{M_0}{2}\hat{z}\left[\text{sign}(z-L) - \text{sign}(z) + \frac{z}{\sqrt{a^2+z^2}} + \frac{L-z}{\sqrt{a^2+(L-z)^2}}\right]. \tag{5.19-5}$$

Note that $\text{sign}(z-L) - \text{sign}(z) = -2\Theta(L-z)\Theta(z)$ (only inside cylinder), so

$$\boxed{\mathbf{H}(0,z) = -\mathbf{M} + \frac{M_0}{2}\hat{z}\left[\frac{z}{\sqrt{a^2+z^2}} + \frac{L-z}{\sqrt{a^2+(L-z)^2}}\right]} \tag{5.19-6}$$

where I have rediscovered the previous expression for the magnetization, $\mathbf{M} = M_0\hat{z}\Theta(a-s)\Theta(z)\Theta(L-z)$, noting that I need not a radial Heaviside function since this is evaluated on the axis. Since $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$, then

$$\boxed{\mathbf{B}(0,z) = \frac{\mu_0 M_0}{2}\hat{z}\left[\frac{z}{\sqrt{a^2+z^2}} + \frac{L-z}{\sqrt{a^2+(L-z)^2}}\right]} \tag{5.19-7}$$

is the magnetic induction.

(b) Plot the ratios $\mathbf{B}/\mu_0 M_0$ and $\mathbf{H}/M_0$ on the axis as functions of z for $L/a = 5$.

Solution: Below is my plot of the ratios, Note that I am superimposing the two graphs, and the $\mathbf{B}$ graph still continues

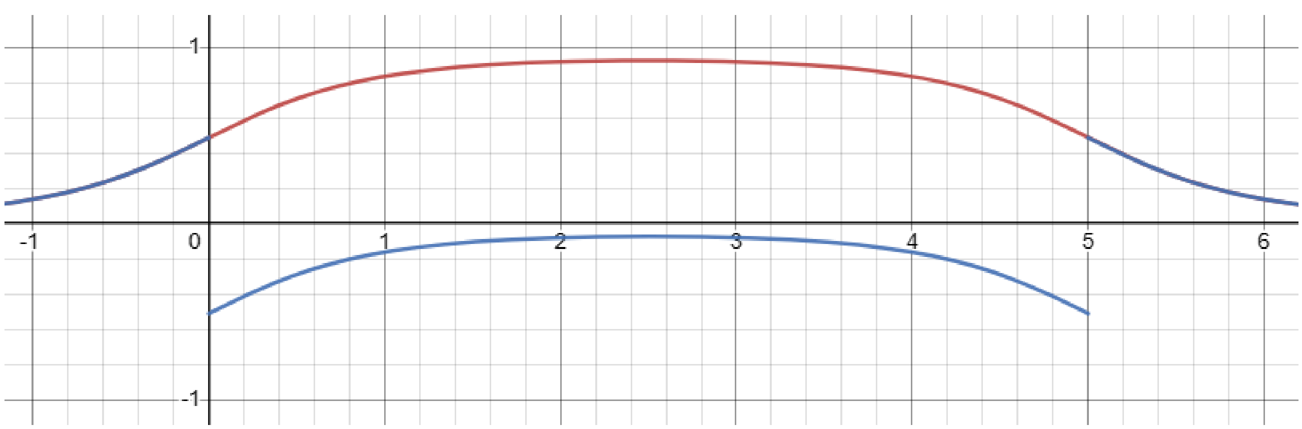


Figure 7: $\mathbf{B}/\mu_0 M_0$ (red) and $\mathbf{H}/M_0$ (blue) on cylinder axis where $L = 5a$ and z is measured in units of a.

outside the cylinder. Clearly, the magnetic induction is continuous at the edges of the cylinder, but the magnetic field $\mathbf{H}$ suffers a discontinuity (since it bears the magnetization explicitly). This discontinuity is a manifestation of the magnetic surface charge on the ends of the cylinder.

Jackson (5.29)

The figure represents a transmission line consisting of two, parallel perfect conductors of arbitrary, but constant, cross section. Current flows down one conductor and returns via the other.

Show that the product of the inductance per unit length L and the capacitance per unit length C is

$$LC = \mu\epsilon \tag{5.29-1}$$

where μ, ϵ are the permeability and the permittivity of the medium surrounding the conductors. (See the discussion about magnetic fields near perfect conductors at the beginning of section 5.13).

Solution: Let the transmission line have length l and surface area S. The idea is to put charges $\pm Q$ on either surface to determine the capacitance, then to put currents $\pm I$ on either surface to determine the inductance. The charge density consistent with a charge $\pm Q$ on each conductor has the form

$$\rho(\mathbf{r}) = \int_S da' \sigma(\mathbf{r}')\delta^3(\mathbf{r} - \mathbf{r}'), \tag{5.29-2}$$

which implies that its corresponding potential has the form

$$\Phi(\mathbf{r}) = \int_V d^3r' \, \frac{\rho(\mathbf{r}')}{4\pi\epsilon||\mathbf{r} - \mathbf{r}'||} = \frac{1}{4\pi\epsilon} \int_S da' \, \frac{\sigma(\mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||}, \tag{5.29-3}$$

where I used δ^3 to collapse the volume integration. It follows, then, that the energy of the charge configuration has the form

$$\begin{aligned} W_e = \frac{1}{2}\int_V d^3r \rho(\mathbf{r})\Phi(\mathbf{r}') &= \frac{1}{8\pi\epsilon}\int_V d^3r \delta^3(\mathbf{r} - \mathbf{r}')\int_S da \int_S da' \frac{\sigma(\mathbf{r})\sigma(\mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||} \\ &= \frac{1}{8\pi\epsilon}\int_S da \int_S da' \, \frac{\sigma(\mathbf{r})\sigma(\mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||}. \end{aligned} \tag{5.29-4}$$

Since this energy is manifest through the alternative form $W_e = \frac{1}{2}\frac{Q^2}{C}$, such that

$$\frac{1}{8\pi\epsilon}\int_S da \int_S da' \, \frac{\sigma(\mathbf{r})\sigma(\mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||} = \frac{Q^2}{2C}. \tag{5.29-5}$$

Now the current density consistent with a current $\pm I$ on each conductor has the analogous form

$$\mathbf{J}(\mathbf{r}) = \int_S da' \, \mathbf{K}(\mathbf{r})\delta^3(\mathbf{r} - \mathbf{r}'), \tag{5.29-6}$$

which implies the corresponding vector potential has the form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||} = \frac{\mu}{4\pi}\int_S da' \, \frac{\mathbf{K}(\mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||}. \tag{5.29-7}$$

The energy of this current configuration then has the form

$$\begin{aligned} W_m = \frac{1}{2}\int_V d^3r \mathbf{J}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}') &= \frac{\mu}{8\pi}\int_V d^3r \delta^3(\mathbf{r} - \mathbf{r}')\int_S da \int_S da' \, \frac{\mathbf{K}(\mathbf{r}) \cdot \mathbf{K}(\mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||} \\ &= \frac{\mu}{8\pi}\int_S da \int_S da' \, \frac{\mathbf{K}(\mathbf{r}) \cdot \mathbf{K}(\mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||} \end{aligned} \tag{5.29-8}$$

Since this energy is manifest in the alternative form $W_m = \frac{1}{2}LI^2$, I find the relation

$$\frac{\mu}{8\pi}\int_S da \int_S da' \, \frac{\mathbf{K}(\mathbf{r}) \cdot \mathbf{K}(\mathbf{r}')}{||\mathbf{r} - \mathbf{r}'||} = \frac{1}{2}LI^2. \tag{5.29-9}$$

Both $\sigma(\mathbf{r})$ and $\mathbf{K}(\mathbf{r})$ distribute themselves in a manner to minimize W such that (depending on the case) the conductors hold charges $\pm Q$ or currents $\pm I$ respectively. Hence, $\sigma(\mathbf{r})$ and $\mathbf{K}(\mathbf{r})$ obey the same minimization problems, and must differ by a constant, $\mathbf{K}(\mathbf{r}) = \alpha\sigma(\mathbf{r})\hat{z}$. Clearly, this constant must have dimensions of velocity, and can be determined without assuming the geometry of the problem using a simple argument,

$$Il = \int da' \; K(\mathbf{r}') = \alpha \int da' \sigma(\mathbf{r}'), \tag{5.29-10}$$

but since $\int da' \sigma(\mathbf{r}') = Q$, I find that $\alpha = Il/Q$. Substituting this choice, $\mathbf{K}(\mathbf{r}) = \frac{Il}{Q}\sigma(\mathbf{r})\hat{z}$ into (5.29-9):

$$\frac{\mu}{8\pi}\frac{I^2 l^2}{Q^2}\int_S da \int_S da' \; \frac{\sigma(\mathbf{r})\sigma(\mathbf{r}')}{||\mathbf{r}-\mathbf{r}'||} = \frac{1}{2}LI^2, \tag{5.29-11}$$

and substituting (5.29-5) in for the integral,

$$\begin{gathered} \frac{\mu}{8\pi}\frac{I^2 l^2}{Q^2}\left(\frac{4\pi\epsilon Q^2}{C}\right) = \frac{1}{2}LI^2 \\ \mu\epsilon = \frac{L}{l}\frac{C}{l} \implies \boxed{L_l C_l = \mu\epsilon} \end{gathered} \tag{5.29-12}$$

Therefore, the product of the capacitance per unit length C_l and the inductance per unit length L_l is equal to $\mu\epsilon$.

Jackson (5.30)

(a) Show that a surface current density $K(\phi) = I\cos(\phi)/2R$ flowing in the axial direction on a right circular cylindrical surface of radius R produces inside the cylinder a uniform magnetic induction $B_0 = \mu_0 I/4R$ in a direction perpendicular to the cylinder axis. Show that the field outside is that of a 2d dipole.

Solution: There is no current in space except for $r = R$, so I can defined a magnetic scalar potential $\nabla^2\Phi_m$ for all regions except for that discontinuity. Since the source involves $\cos\phi$, I only need $m = 1$ in the general solution to Laplace's equation in cylindrical coordinates, so guess that

$$\Phi_m(s,\phi) = \begin{cases} \alpha s \sin\phi, & s < R \\ \frac{\beta}{s}\sin\phi, & s > R \end{cases} \tag{5.30-1}$$

The associated magnetic fields are

$$H_s = -\hat{s}\partial_s\Phi_m = \begin{cases} -\alpha\sin\phi, & s < R \\ \frac{\beta}{s^2}\sin\phi, & s > R \end{cases}, \qquad H_\phi = -\hat{\phi}\frac{1}{s}\partial_\phi\Phi_m = \begin{cases} -\alpha\cos\phi, & s < R \\ -\frac{\beta}{s^2}\cos\phi, & s > R \end{cases} \tag{5.30-2}$$

which obey the BC's $\hat{s}\cdot\Delta\mathbf{B}\Big|_{r=R} = 0$ and $\hat{s}\times\Delta\mathbf{H}\Big|_{r=R} = K\hat{z}$. The second BC picks out only H_ϕ, so I find two equations

$$\alpha + \frac{\beta}{R^2} = 0, \qquad \alpha - \frac{\beta}{R^2} = \frac{I}{2R}, \tag{5.30-3}$$

which determine $\alpha = I/4R$ and $\beta = -IR/4$. Therefore, the magnetic field (5.30-2) is

$$\mathbf{H}_{\text{in}} = -\alpha\left(\sin\phi\hat{s} + \cos\phi\hat{\phi}\right) = -\frac{I}{4R}\hat{y} \implies \boxed{|\mathbf{B}_{\text{in}}| \equiv B_0 = \frac{\mu_0 I}{4R}} \tag{5.30-4}$$

and thus inside the cylinder is a uniform magnetic induction $B_0 = \mu_0 I/4R$, as desired. The field outside is

$$\mathbf{H}_{\text{out}} = \frac{\beta}{s^2}\left(\sin\phi\hat{s} - \cos\phi\hat{\phi}\right) = -\frac{IR}{4s^2}\left(\sin\phi\hat{s} - \cos\phi\hat{\phi}\right), \tag{5.30-5}$$

and through this structure is indeed a 2-dimensional dipole.

(b) Calculate the total magnetostatic field energy per unit length. How is it divided inside and outside the cylinder?

Solution: The energy densities in each region are

$$\frac{1}{2}\mathbf{H}_{\text{in}}\cdot\mathbf{B}_{\text{in}} = \frac{\mu_0 I^2}{16R^2}, \qquad \frac{1}{2}\mathbf{H}_{\text{out}}\cdot\mathbf{B}_{\text{out}} = \frac{\mu_0 I^2 R^2}{16 s^4}, \tag{5.30-6}$$

such that the magnetostatic energy per unit length in each region is

$$\frac{\partial W_{\text{in}}}{\partial z} = \frac{1}{2}(2\pi)\int_0^R ds s\frac{\mu_0 I^2}{16R^2} = \frac{\pi\mu_0 I^2}{32}, \qquad \frac{\partial W_{\text{out}}}{\partial z} = \frac{1}{2}(2\pi)\int_0^R ds s\frac{\mu_0 I^2 R^2}{16 s^4} = \frac{\pi\mu_0 I^2}{32}, \tag{5.30-7}$$

and the magnetostatic energy per unit length is divided evenly between each region. This gives a total magnetostatic energy per unit length of

$$\boxed{\frac{\partial W}{\partial z} = \frac{\pi\mu_0 I^2}{16}} \tag{5.30-8}$$

(c) What is the inductance per unit length of the system, viewed as a long circuit with current flowing up one side of the cylinder and back the other? (Answer: $L = \pi\mu_0/8$).

Solution: The inductance per unit length L_l can be found from $\frac{\partial W}{\partial z} = \frac{1}{2}L_l I^2$, where using (5.30-8) immediately gives

$$\frac{1}{2}L_l I^2 = \frac{\pi\mu_0 I^2}{16} \implies \boxed{L_l = \frac{\pi\mu_0}{8}} \tag{5.30-9}$$

as given in the hint to the problem.

Jackson (5.31)

An accelerator bending magnet consists of N turns of superconducting cable whose current configuration can be described approximately by the axial current density

$$J_z(\rho,\phi) = \left(\frac{NI}{2R}\right)\cos\phi\,\delta(\rho - R). \tag{5.31-1}$$

The right circular current cylinder is centered on the axis of a hollow iron cylinder of inner radius $R' > R$. The relative dimensions (R, R' a few centimeters and a magnet length of several meters) permit the use of a two-dimensional approximation, at least away from the ends of the magnet. Assume that the relative permeability of the iron can be taken as infinite. (Then the outer radius of the iron is irrelevant.)

(a) Show that the magnetic field inside the current sheath is perpendicular to the axis of the cylinder in the direction defined by $\phi = \pm\pi/2$ and has the magnitude

$$B_0 = \left(\frac{\mu_0 NI}{4R}\right)\left[1 + \frac{R^2}{R'^2}\right] \tag{5.31-2}$$

Solution: By "inside the current sheath" Jackson means for $0 < s < R$. In matching boundary conditions it is convenient to integrate the axial current density (5.31-1), using the delta function to map it to a surface current,

$$\mathbf{K}(\mathbf{r}) = \frac{NI}{2R}\cos\phi\,\hat{z}. \tag{5.31-3}$$

Since $J_{\text{free}} = 0$ for all of space, it is true that $\nabla\times\mathbf{H} = 0 \implies \mathbf{H} = -\nabla\Phi$ and the magnetic field can be written as the gradient of a scalar potential:

$$\mathbf{H} = -\nabla\Phi_m = -\hat{s}\partial_s\Phi - \frac{\hat{\phi}}{s}\partial_\phi\Phi \equiv H_s\hat{s} + H_\phi\hat{\phi}. \tag{5.31-4}$$

Since the relative permeability of iron can be taken as infinite, it forces us to take the magnetic field in the iron to vanish, $\mathbf{H}_{s>R'} = 0$. The boundary conditions at $s = R$ are

$$\begin{aligned} \hat{s}\cdot\Delta\mathbf{B} = 0 &\implies H_{s,\text{out}}\Big|_R = H_{s,\text{in}}\Big|_R \\ \hat{s}\times\Delta\mathbf{H} = \mathbf{K} &\implies H_{\phi,\text{out}}\Big|_R - H_{\phi,\text{in}}\Big|_R = \frac{NI}{2R}\cos\phi, \end{aligned} \tag{5.31-5}$$

where since $\mathbf{K}\parallel\hat{z}$ the second boundary condition picks out only the angular dependence of the magnetic field. In the first boundary condition note that I free replaced $B\to H$ since the permeability is vacuum inside and outside. Since the permeability of iron is infinite, at $s = R'$ only the boundary condition parallel to the surface is valid:

$$\hat{s}\times\Delta\mathbf{H} = \mathbf{K} \implies \cancel{H_{\phi,\text{out}}}\Big|_{R'} - H_{\phi,\text{in}}\Big|_{R'} = 0 \implies H_{\phi,\text{in}}\Big|_{R'} = 0. \tag{5.31-6}$$

These give a set of three equations that determine the coefficients of the magnetic field in each region. Since there is no free charge/currents in the two regions $0 < s < R$ and $R < s < R'$, it suffices to make the usual expansions

$$\Phi = \begin{cases} \alpha s\sin\phi, & 0 < s < R \\ \left(\beta s + \frac{\gamma}{s}\right)\sin\phi, & R < s < R' \end{cases} \tag{5.31-7}$$

for the scalar potential. I chose only the sin component because the source of all the fields in this problem (5.31-3) involves only a $\cos\phi$. The radial terms are the usual factors of $as^m + bs^{-m}$ under the choice $m = 1$ to get only a $\sin\phi$ dependence. Evaluating the necessary derivatives,

$$H_s = \begin{cases} -\alpha\sin\phi, & 0 < s < R \\ -\left(\beta - \frac{\gamma}{s^2}\right)\sin\phi, & R < s < R' \end{cases}, \qquad H_\phi = \begin{cases} -\alpha\cos\phi, & 0 < s < R \\ -\left(\beta + \frac{\gamma}{s^2}\right)\cos\phi, & R < s < R' \end{cases} \tag{5.31-8}$$

allows me to write the three boundary conditions (5.31-5) and (5.31-6) as three equations in three unknowns,

$$\begin{aligned} \alpha - \beta &= -\frac{\gamma}{R^2} \\ -\alpha + \beta + \frac{\gamma}{R^2} &= -\frac{NI}{2R} \\ \beta &= -\frac{\gamma}{R'^2} \end{aligned} \tag{5.31-9}$$

Which can be easily solved (add the first two then plug in the third to get γ, from which all else follows):

$$\gamma = -\frac{NIR}{4}, \qquad \beta = \frac{NIR}{4R'^2}, \qquad \alpha = \frac{NI}{4R}\left(1 + \frac{R^2}{R'^2}\right). \tag{5.31-10}$$

It follows, then, that the magnetic field inside the current sheath has the form

$$\begin{aligned} \mathbf{H}_{0<s<R} &= -\alpha \sin\phi \hat{s} - \alpha \cos\phi \hat{\phi} \\ &= -\frac{NI}{4R}\left(1 + \frac{R^2}{R'^2}\right)\left(\sin\phi \hat{s} + \cos\phi \hat{\phi}\right) \\ &= -\frac{NI}{4R}\left(1 + \frac{R^2}{R'^2}\right)\hat{y}. \end{aligned} \tag{5.31-11}$$

Multiplying both sides by μ_0, I find that

$$\boxed{\mathbf{B}_{0<s<R} = -\frac{\mu_0 NI}{4R}\left(1 + \frac{R^2}{R'^2}\right)\hat{y}} \tag{5.31-12}$$

which is the desired relation (5.31-2) in the problem statement. The $\hat{y}$ vector indeed aligns the (constant) magnetic field in the direction defined by $\phi = \pm\pi/2$.

(b) Show that the magnetic energy inside $r = R$ is augmented (and that outside diminished) relative to the values in the absence of the iron. (Compare part b of Problem 5.30).

Solution: The magnetic energy density inside $s = R$ is given by

$$\mathbf{B}\cdot\mathbf{H} = \frac{1}{\mu_0}|\mathbf{B}|^2 = \frac{\mu_0 N^2 I^2}{16R^2}\left(1+\frac{R^2}{R'^2}\right)^2 . \tag{5.31-13}$$

In the absence of iron, the magnetic energy (per unit length) inside has the form $W/l = \pi\mu_0 I^2/32$. The corresponding magnetic energy per unit length to (5.31-13) is given by

$$W_{\text{in}} = \frac{1}{2}\int d^3r'\mathbf{B}\cdot\mathbf{H} = \frac{\mu_0 N^2 I^2}{32R^2}\left(1+\frac{R^2}{R'^2}\right)^2\int d^3r', \tag{5.31-14}$$

$$\boxed{\frac{W_{\text{in}}}{l} = \frac{\pi\mu_0 N^2 I^2}{32}\left(1+\frac{R^2}{R'^2}\right)^2} \tag{5.31-15}$$

where l is the length. Since for problem 5.30 there is no N in the free surface current, I can set $N = 1$ in (5.31-15) and directly compare the two relations. Since here $(1 + R^2/R'^2)^2 > 0$, then the energy (5.31-15) in the presence of the iron is larger (augmented) relative to the energy without the iron. In the outside region $R < s < R'$, the magnetic field has the form

$$\mathbf{H}_{R<s<R'} = -\left(\beta - \frac{\gamma}{s^2}\right)\sin\phi\hat{s} - \left(\beta + \frac{\gamma}{s^2}\right)\cos\phi\hat{\phi} \tag{5.31-16}$$

such that the magnetic energy density is

$$\begin{aligned}\mathbf{B}\cdot\mathbf{H} = \mu_0\left[\left(\beta-\frac{\gamma}{s^2}\right)^2\sin^2\phi + \left(\beta+\frac{\gamma}{s^2}\right)^2\cos^2\phi\right] &= \mu_0\left[\beta^2 + \frac{\gamma^2}{s^4} + \frac{2\beta\gamma}{s^2}\left(\cos^2\phi - \sin^2\phi\right)\right]\\ &= \mu_0\left[\beta^2 + \frac{\gamma^2}{s^4} + \frac{2\beta\gamma}{s^2}\cos(2\phi)\right],\end{aligned} \tag{5.31-17}$$

which can be integrated to give

$$\begin{aligned} W_{\text{out}} &= \frac{\mu_0}{2}(l)\int_0^{2\pi} d\phi\int_R^{R'}\left[\beta^2 s + \frac{\gamma^2}{s^3} + \cancel{\frac{2\beta\gamma}{s}\cos(2\phi)}\right]\\ \therefore\ \frac{W_{\text{out}}}{l} &= \frac{\pi\mu_0}{2}\left[\beta^2(R'^2 - R^2) - \gamma^2\left(\frac{1}{R'^2} - \frac{1}{R^2}\right)\right]\\ &= \frac{\pi\mu_0 N^2 I^2}{32}\left[\frac{R^2R'^4}{R^2R'^4} - \frac{R^6}{R^2R'^4}\right] \implies \boxed{\frac{W_{\text{out}}}{l} = \frac{\pi\mu_0 N^2 I^2}{32}\left[1 - \frac{R^4}{R'^4}\right]}\end{aligned} \tag{5.31-18}$$

Hence, the energy outside is diminished relative to the energy without the iron $\pi\mu_0 I^2/32$, as $1 - R^4/R'^4 < 1$.
(c) Show that the inductance per unit length is

$$\frac{dL}{dz} = \left(\frac{\pi\mu_0 N^2}{8}\right)\left[1 + \frac{R^2}{R'^2}\right]$$

Solution: The total magnetic energy per unit length is the sum of (5.31-15) and (5.31-18),

$$\frac{W}{l} = \frac{\pi\mu_0 N^2 I^2}{32}\left[\left(1+\frac{R^2}{R'^2}\right)^2 + 1 - \frac{R^4}{R'^4}\right] = \frac{\pi\mu_0 N^2 I^2}{16}\left[1 + \frac{R^2}{R'^2}\right] \tag{5.31-19}$$

Since magnetic energy can be identified as the quantity $\frac{1}{2}LI^2$, it follows that

$$\frac{1}{2}\frac{L}{l}I^2 = \frac{\pi\mu_0 N^2 I^2}{16}\left[1 + \frac{R^2}{R'^2}\right] \implies \boxed{\frac{L}{l} = \frac{\pi\mu_0 N^2}{8}\left[1 + \frac{R^2}{R'^2}\right]} \tag{5.31-20}$$

Chapter 6

Jackson (6.1)

In 3d the solution to the wave equation (6.32) for a point source in space and time (a light flash at $t' = 0$, $\mathbf{x}' = 0$) is a spherical shell disturbance of radius $R = ct$, namely the Green's function $G^{(+)}$ (6.44). It may be initially surprising that in one or two dimensions, the disturbance possesses a "wake", even though the source is a "point" in space and time. The solutions for fewer dimensions than three can be found by superposition in the superfluous dimension(s), to eliminate dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.

(a) Starting with the retarded solution to the 3d wave equation (6.47), show that the source $f(\mathbf{x}', t') = \delta(x')\delta(y')\delta(t')$, equivalent to a $t = 0$ point source at the origin in two spatial dimensions, produces a two-dimensional wave

$$\Psi(x, y, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}} \tag{6.1-1}$$

where $\rho^2 = x^2 + y^2$ and $\Theta(\xi)$ is the unit step function.

Solution: The retarded solution to the 3d wave equation is

$$\Psi(\mathbf{x}, t) = \int d^3x' \, \frac{[f(\mathbf{x}', t')]_{\text{ret}}}{||\mathbf{x} - \mathbf{x}'||}, \tag{6.1-2}$$

where since $[f(\mathbf{x}', t')]_{\text{ret}} = \delta(x')\delta(y')\delta(t - \frac{1}{c}||\mathbf{x} - \mathbf{x}'||)$, I find

$$\begin{aligned}
\Psi(\mathbf{x}, t) &= \int d^3x' \, \delta(x')\delta(y') \frac{\delta(t - \frac{1}{c}||\mathbf{x} - \mathbf{x}'||)}{||\mathbf{x} - \mathbf{x}'||} \\
&= \int d^3x' \, \delta(x')\delta(y') \frac{\delta(t - \frac{1}{c}\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2})}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \\
&= \int dz' \, \frac{\delta(t - \frac{1}{c}\sqrt{\rho^2 + (z - z')^2})}{\sqrt{\rho^2 + (z - z')^2}}.
\end{aligned} \tag{6.1-3}$$

I will expand the delta function $\delta(g(z'))$. Since $g(z') = 0$ gives the roots

$$z - z'_{\pm} = \pm\sqrt{c^2t^2 - \rho^2} \tag{6.1-4}$$

and the derivative $g'(z')$ evaluated at these roots is

$$|g'(z'_{\pm})| = \left| \frac{z - z'}{c\sqrt{\rho^2 + (z - z')^2}} \right|_{z' = z - z'_{\pm}} = \frac{1}{c^2t}\sqrt{c^2t^2 - \rho^2}, \tag{6.1-5}$$

then I find that

$$\delta\left(t - \frac{1}{c}\sqrt{\rho^2 + (z - z')^2}\right) = \sum_{i=\pm} \frac{\delta(z' - z'_i)}{|g'(z'_i)|} = c^2t \cdot \frac{\delta(z' - z'_+) + \delta(z' - z'_-)}{\sqrt{c^2t^2 - \rho^2}}. \tag{6.1-6}$$

If follows that (6.1-3) can be written as

$$\begin{aligned}
\Psi(\mathbf{x}, t) &= \frac{c^2t}{\sqrt{c^2t^2 - \rho^2}} \int dz' \, \frac{\delta(z' - z'_+) + \delta(z' - z'_-)}{\sqrt{\rho^2 + (z - z')^2}} \\
&= \frac{c^2t}{\sqrt{c^2t^2 - \rho^2}} \left[\frac{\Theta(ct - \rho)}{\sqrt{\rho^2 + (z - z'_+)^2}} + \frac{\Theta(ct - \rho)}{\sqrt{\rho^2 + (z - z'_-)^2}} \right] \\
&= \frac{c^2t}{\sqrt{c^2t^2 - \rho^2}} \left[\frac{2\Theta(ct - \rho)}{ct} \right].
\end{aligned} \tag{6.1-7}$$

where the integral is only nonzero if the integration measure hits both $z'_\pm$, resulting in a Heaviside function. Therefore,

$$\boxed{\Psi(x,y,t) = \frac{2c\Theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}} \tag{6.1-8}$$

as desired.

(b) Show that a "sheet" source, equivalent to a point pulsed source at the origin in one space dimension, produces a one-dimensional wave proportional to

$$\Psi(x,t) = 2\pi c\Theta(ct-|x|), \tag{6.1-9}$$

Solution: I will play the same game. Here, a point pulsed source at the origin $x'=0$ in on spatial dimension has the form of $f(x',t') = \delta(x')\delta(t')$. This implies a retarded form $[f(x',t')]_{\text{ret}} = \delta(x')\delta(t-\frac{1}{c}||\mathbf{x}-\mathbf{x}'||)$, such that (6.1-2) can be written as

$$\begin{aligned}\Psi(\mathbf{x},t) &= \int d^3x'\delta(x')\frac{\delta\left(t-\frac{1}{c}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\\ &= \int dy'dz'\frac{\delta\left(t-\frac{1}{c}\sqrt{x^2+(y-y')^2+(z-z')^2}\right)}{\sqrt{x^2+(y-y')^2+(z-z')^2}}\end{aligned} \tag{6.1-10}$$

Let $\tilde{y}=y-y'$ and $\tilde{z}=z-z'$, defining $\tilde{\rho}^2=\tilde{y}^2+\tilde{z}^2$ in the process,

$$\begin{aligned}\Psi(\mathbf{x},t) &= \int d\tilde{\rho}d\phi\ \frac{\tilde{\rho}}{\sqrt{x^2+\tilde{\rho}^2}}\delta\left(t-\frac{1}{c}\sqrt{x^2+\tilde{\rho}^2}\right)\\ &= 2\pi\int d\tilde{\rho}\ \frac{\tilde{\rho}}{\sqrt{x^2+\tilde{\rho}^2}}\delta\left(t-\frac{1}{c}\sqrt{x^2+\tilde{\rho}^2}\right),\end{aligned} \tag{6.1-11}$$

where I made the natural transformation to polar coordinates and subsequently integrated over all angles. I will expand the delta function $\delta(g(\tilde{\rho}))$. Technically, $g(\tilde{\rho}')=0$ has two roots, but since radii are positive I will take only a single root:

$$\tilde{\rho}_+ = \sqrt{c^2t^2-x^2}. \tag{6.1-12}$$

The derivative $g'(\tilde{\rho}')$ evaluated at this root is

$$|g'(\tilde{\rho}_+)| = \frac{\tilde{\rho_+}}{c\sqrt{c^2+\tilde{\rho}^2}} = \frac{\sqrt{c^2t^2-x^2}}{c^2t}, \tag{6.1-13}$$

so the delta function can be written as

$$\delta\left(t-\frac{1}{c}\sqrt{x^2+\tilde{\rho}^2}\right) = \frac{c^2t}{\sqrt{c^2t^2-x^2}}\delta(\tilde{\rho}-\tilde{\rho}_+), \tag{6.1-14}$$

and (6.1-11) can be written as

$$\begin{aligned}\Psi(\mathbf{x},t) &= \frac{2\pi c^2t}{\sqrt{c^2t^2-x^2}}\int d\tilde{\rho}\ \frac{\tilde{\rho}}{\sqrt{x^2+\tilde{\rho}^2}}\delta(\tilde{\rho}-\tilde{\rho}_+)\\ &= \frac{2\pi c^2t}{\sqrt{c^2t^2-x^2}}\frac{\tilde{\rho}_+\Theta(ct-|x|)}{\sqrt{x^2+\tilde{\rho}_+^2}}\\ &= \frac{2\pi c^2t\Theta(ct-|x|)}{\sqrt{c^2t^2-x^2}}\frac{\sqrt{c^2t^2-x^2}}{ct},\end{aligned} \tag{6.1-15}$$

where again the integral is only nonzero if it hits the value $\tilde{\rho}_+$, resulting in a Heaviside function. Thus,

$$\boxed{\Psi(x,t) = 2\pi c\Theta(ct-|x|)} \tag{6.1-16}$$

as desired.

Jackson (6.5)

A localized charge distribution produces an electrostatic field, $\mathbf{E} = -\nabla\Phi$. Into this field is place a small localized time-independent current density $\mathbf{J}(\mathbf{r})$, which generates a magnetic field $\mathbf{H}$.

(a) Show that the momentum of these electromagnetic fields, (6.117), can be transformed to

$$\mathbf{P}_{\text{field}} = \frac{1}{c^2}\int d^3r\ \Phi\mathbf{J},$$

provided the product $\Phi\mathbf{H}$ falls off rapidly enough at large distances. How rapidly is "rapidly enough"?

Solution: Using the general formula,

$$\begin{aligned}\mathbf{P}_{\text{field}} = \mu_0\epsilon_0\int_V d^3r\ \mathbf{E}\times\mathbf{H} &= \frac{1}{c^2}\int_V d^3r\ \mathbf{H}\times(\nabla\Phi)\\ &= \frac{1}{c^2}\epsilon^{ijk}\int_V d^3r\ H^j\frac{\partial\Phi}{\partial x^k}.\end{aligned} \tag{6.5-1}$$

Performing integration by parts,

$$\begin{aligned}\mathbf{P}_{\text{field}} &= \frac{1}{c^2}\epsilon^{ijk}\left[\int_{\partial V} da_k H^j\Phi - \int_V d^3r\ \frac{\partial H^j}{\partial x^k}\Phi\right]\\ &= \frac{1}{c^2}\epsilon^{ijk}\left[\int_{\partial V} d\Omega\ \hat{n}^k(r^2H^j\Phi) - \int_V d^3r\ \frac{\partial H^j}{\partial x^k}\Phi\right],\end{aligned} \tag{6.5-2}$$

where $\hat{n}$ is the outward normal vector to the oriented surface ∂V. We are given that $\Phi\mathbf{H}$ falls off rapidly at large distances, and it needs to fall off rapidly enough that the quantity $r^2H^j\Phi \to 0$ as $r \to 0$, so that the surface integral vanishes ($H\Phi \to 0$ faster than $r^2 \to \infty$). This condition leads to

$$\mathbf{P}_{\text{field}} = \frac{1}{c^2}\int_V d^3r\ (\nabla\times\mathbf{H})\Phi, \tag{6.5-3}$$

where since we have an electrostatic field, $\nabla\times\mathbf{H} = \mathbf{J}$, as the time derivative of the electric displacement vanishes. Thus,

$$\boxed{\mathbf{P}_{\text{field}} = \frac{1}{c^2}\int_V d^3r\ \Phi\mathbf{J}} \tag{6.5-4}$$

as desired.

(b) Assuming that the current distribution is localized to a region small compared to the scale of variation of the electric field, expand the electrostatic potential in a Taylor series and show that

$$\mathbf{P}_{\text{field}} = \frac{1}{c^2}\mathbf{E}(0)\times\mathbf{m}, \tag{6.5-5}$$

where $\mathbf{E}(0)$ is the electric field at the current distribution and $\mathbf{m}$ is the magnetic moment, (5.54), caused by the current.

Solution: Expanding the potential in a Taylor series to second order, $\Phi(\mathbf{r}) = \Phi(0) - \mathbf{E}(0)\cdot\mathbf{r}$, which implies that (6.5-4) becomes

$$\begin{aligned}\mathbf{P}_{\text{field}} &= \frac{1}{c^2}\Phi(0)\cancel{\int_V d^3r}\ \mathbf{J} - \frac{1}{c^2}\int_V d^3r\ (\mathbf{E}(0)\cdot\mathbf{r})\mathbf{J}\\ &= -\frac{1}{c^2}\int_V d^3r\ (\mathbf{E}(0)\cdot\mathbf{r})\mathbf{J},\end{aligned} \tag{6.5-6}$$

where I used the fact that the current distribution is localized to a region small to the variation scale of $\mathbf{E}$, so that its integral over all of space of the current can be neglected. Using index notation and the identity $\int d^3r\ r^i J^j = \epsilon^{ijk} m^k$, I find

$$\begin{aligned} P^j_{\text{field}} &= -\frac{1}{c^2}\mathbf{E}^i(0)\int_V d^3r\ r^i J^j = -\frac{1}{c^2}\epsilon^{ijk}\mathbf{E}^i(0)m^k \\ &= \frac{1}{c^2}\epsilon^{jik}\mathbf{E}^i(0)m^k \implies \boxed{\mathbf{P}_{\text{field}} = \frac{1}{c^2}\mathbf{E}(0)\times\mathbf{m}} \end{aligned} \tag{6.5-7}$$

as desired.

(c) Suppose the current distribution is placed instead in a uniform electric field $\mathbf{E}_0$ (filling all space). Show that, no matter how complicated is the localized $\mathbf{J}$, the result in part (a) is augmented by a surface integral contribution from infinity equal to minus one-third of the result of part (b), yielding

$$\mathbf{P}_{\text{field}} = \frac{2}{3c^2}\mathbf{E}(0)\times\mathbf{m}. \tag{6.5-8}$$

Compare this result with that obtained by working directly with (6.117) and the considerations at the end of Section 5.6.

Solution: Not neglecting the surface integral and using my result (6.5-7) for the non-vanishing term, I find in general that

$$\mathbf{P}_{\text{field}} = \frac{1}{c^2}\mathbf{E}(0)\times\mathbf{m} + \frac{1}{c^2}\epsilon^{ijk}\int_{\partial V} d\Omega\ \hat{r}^k(r^2 H^j\Phi). \tag{6.5-9}$$

Jackson says I should get $-1/3$ of the first term listed, and that guy has an magnetic moment $\mathbf{m}$ in it, so it makes to sense to use the following,

$$H^j = \frac{1}{4\pi}\left[\frac{3\hat{r}^j\hat{r}^l m^l - m^j}{r^3}\right] = \frac{3}{4\pi}\frac{\hat{r}^j\hat{r}^l m^l}{r^3} - \frac{m^j}{4\pi r^3}, \tag{6.5-10}$$

which is the lowest non-vanishing contribution to the magnetic field. This is justified because the current distribution is small compared to the variation scale of the electric field, and the electric field is itself uniform. Using the same form of the potential, $\Phi = \Phi(0) - rE^n(0)\hat{r}^n$, then

$$\begin{aligned} \mathbf{P}_{\text{field}} &= \frac{1}{c^2}\mathbf{E}(0)\times\mathbf{m} + \cancel{\frac{3\Phi(0)}{4\pi c^2}\epsilon^{ijk}\int_{\partial V} d\Omega\,\frac{1}{r}\left(3\hat{r}^k\hat{r}^j\hat{r}^l m^l - \hat{r}^k m^j\right)} \\ &\qquad - \frac{E^n(\mathbf{0})}{4\pi c^2}\epsilon^{ijk}\int_{\partial V} d\Omega\ \left(3\hat{r}^k\hat{r}^j\hat{r}^l\hat{r}^n m^l - \hat{r}^k\hat{r}^n m^j\right) \\ &= \frac{1}{c^2}\mathbf{E}(0)\times\mathbf{m} - \frac{E^n(\mathbf{0})}{4\pi c^2}\epsilon^{ijk}\left(\frac{4\pi}{5}\left(\delta^{kj}\delta^{nl} + \delta^{kl}\delta^{jn} + \delta^{kn}\delta^{jl}\right)m^l - \frac{4\pi}{3}\delta^{kn}m^j\right) \\ &= \frac{1}{c^2}\mathbf{E}(0)\times\mathbf{m} - \cancel{\frac{E^n(0)}{5c^2}\left(0 + \epsilon^{inl} + \epsilon^{iln}\right)m^l} - \frac{1}{3c^2}\mathbf{E}(0)\times\mathbf{m} \\ &= \frac{2}{3c^2}\mathbf{E}(0)\times\mathbf{m} \end{aligned} \tag{6.5-11}$$

Here, I used the fact that for such a small current current distribution, $\mathbf{m}$ can be taken to be constant, and odd factors of $\hat{r}$ integrate to zero over a solid angle (the even factors are nonzero and give delta functions). I conclude, then, that

$$\therefore\ \boxed{\mathbf{P}_{\text{field}} = \frac{2}{3c^2}\mathbf{E}(0)\times\mathbf{m}} \tag{6.5-12}$$

as desired.

Jackson (6.15)

If a conductor or semiconductor has current flowing in it because of an applied electric field, and a transverse magnetic field is applied, there develops a component of electric field in the direction orthogonal to both the applied electric field (direction of current flow) and the magnetic field, resulting in a voltage difference between the sides of the conductor. This phenomenon is known as the Hall effect.

(a) Use the known properties of electromagnetic fields under rotations and spatial reflections and the assumption of Taylor series expansions around zero magnetic field strength to show that for an isotropic medium the generalization of Ohm's law, correct to second order in the magnetic field, must have the form

$$\mathbf{E} = \rho_0\mathbf{J} + R(\mathbf{H}\times\mathbf{J}) + \beta_1 H^2\mathbf{J} + \beta_2(\mathbf{H}\cdot\mathbf{J})\mathbf{H}, \tag{6.15-1}$$

where ρ_0 is the resistivity in the absence of the magnetic field and R is called the Hall coefficient.

Solution: The electric field $\mathbf{E}$ should be expanded under $\mathbf{J}$ and $\mathbf{H}$ because they are easily adjusted under experimentation. The idea is to expand the electric field in terms of combinations of these factors, and without any derivatives. This amounts to

$$E^i = \alpha^{ij}J^j + \beta^{ij}H^j + \eta^{ijk}H^jJ^k + \gamma^{ijkl}J^jH^kH^l + \chi^{ijkl}J^jJ^kH^l \tag{6.15-2}$$

for some undetermined tensor coefficients. The only rotationally invariant tensors are the Kronecker delta and the Levi-Civita symbol, so if we demand rotational invariance, each tensor is a multiple of those two:

$$\begin{aligned}E^i &= \alpha\delta^{ij}J^j + \beta\delta^{ij}H^j + \eta\epsilon^{ijk}H^jJ^k + \left(\gamma_1\delta^{ij}\delta^{kl} + \gamma_2\delta^{ik}\delta^{jl} + \gamma_3\delta^{il}\delta^{jk}\right)J^jH^kH^l\\ &\qquad + \left(\chi_1\delta^{ij}\delta^{kl} + \chi_2\delta^{ik}\delta^{jl} + \chi_3\delta^{il}\delta^{jk}\right)J^jJ^kH^l\\ \therefore\ \mathbf{E} &= \alpha\mathbf{J} + \beta\mathbf{H} + \eta(\mathbf{H}\times\mathbf{J}) + \gamma H^2\mathbf{J} + \rho(\mathbf{H}\cdot\mathbf{J})\mathbf{H} + \chi J^2H + \psi(\mathbf{H}\cdot\mathbf{J})\mathbf{J}\end{aligned} \tag{6.15-3}$$

which is already very similar. On the grounds of parity, both $\mathbf{E}$ and $\mathbf{J}$ are odd (vectors), and $\mathbf{H}$ is even (pseudovector). So, all terms on the r.h.s of (6.15-3) must be odd under parity. Clearly, $\beta = \chi = \psi = 0$ since even combinations of current are even under parity. Redefining the remaining coefficients to match Jackson's definitions, I find the form

$$\boxed{\mathbf{E} = \rho_0\mathbf{J} + R(\mathbf{H}\times\mathbf{J}) + \beta_1 H^2\mathbf{J} + \beta_2(\mathbf{H}\cdot\mathbf{J})\mathbf{H}} \tag{6.15-4}$$

as desired.

(b) What about the requirements of time reversal invariance?

Solution: Since $\mathbf{J}$ involves a time derivative, it is odd under time reversal. For $\mathbf{E}$ and $\mathbf{H}$, they both have the opposite behavior as parity, so $\mathbf{E}$ is even while $\mathbf{H}$ is odd. Therefore, the r.h.s of (6.15-4) must be even under time reversal invariance. Automatically, $\rho_0 = \beta_1 = \beta_2 = 0$ since $\mathbf{J}$ is odd and two copies of H are always even. The term $\mathbf{H}\times\mathbf{J}$ is even, so it stays (as anticipated because Jackson gave it the "Hall" coefficient). Thus,

$$\boxed{\mathbf{E} = R(\mathbf{H}\times\mathbf{J})} \tag{6.15-5}$$

is the generalization of Ohm's law for this problem, on the grounds of RPT invariance.

Jackson (6.20)

An example of the preservation of causality and finite speed of propagation in spite of the use of the Coulomb gauge is afforded by a dipole source that is flashed on and off at $t = 0$. The effective charge and current densities are

$$\begin{aligned} \rho(\mathbf{x}, t) &= \delta(x)\delta(y)\delta'(z)\delta(t) \\ J_z(\mathbf{x}, t) &= -\delta(x)\delta(y)\delta(z)\delta'(t). \end{aligned} \tag{6.20-1}$$

where a prime means differentiation with respect to the argument. This dipole is of unit strength and it points in the negative z-direction.

(a) Show that the instantaneous Coulomb potential (6.23) is

$$\Phi(\mathbf{x}, t) = -\frac{1}{4\pi\epsilon_0}\delta(t)\frac{z}{r^3}. \tag{6.20-2}$$

Solution: Equation (6.23) tells me that

$$\begin{aligned} \Phi(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0}\int d^3x'\ \frac{\rho(x', t)}{||\mathbf{x} - \mathbf{x}'||} \\ &= \frac{1}{4\pi\epsilon_0}\delta(t)\int d^3x'\ \frac{\delta(x')\delta(y')\delta'(z')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\ &= \frac{1}{4\pi\epsilon_0}\delta(t)\int dz'\ \frac{\delta'(z')}{\sqrt{x^2 + y^2 + (z-z')^2}}, \end{aligned} \tag{6.20-3}$$

which can be simplified by an IBP:

$$\begin{aligned} \Phi(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0}\delta(t)\left[\cancel{\left.\frac{\delta(z')}{\sqrt{x^2 + y^2 + (z-z')^2}}\right|_{z'=-\infty}^{z'=\infty}} - \int dz'\ \delta(z')\frac{\partial}{\partial z'}\left(\frac{1}{\sqrt{x^2 + y^2 + (z-z')^2}}\right)\right] \\ &= -\frac{1}{4\pi\epsilon_0}\delta(t)\int dz'\ \delta(z')\frac{z - z'}{(x^2 + y^2 + (z-z')^2)^{3/2}} \\ &= -\frac{1}{4\pi\epsilon_0}\delta(t)\frac{z}{(x^2 + y^2 + z^2)^{3/2}}. \end{aligned} \tag{6.20-4}$$

Since $r = \sqrt{x^2 + y^2 + z^2}$ then the instantaneous Coulomb potential is

$$\boxed{\Phi(\mathbf{x}, t) = -\frac{1}{4\pi\epsilon_0}\delta(t)\frac{z}{r^3}} \tag{6.20-5}$$

as desired.

(b) Show that the transverse current $\mathbf{J}_t$ is

$$\mathbf{J}_t(\mathbf{x},t) = -\delta'(t)\left[\frac{2}{3}\hat{z}\delta(\mathbf{x}) - \frac{\hat{z}}{4\pi r^3} + \frac{3}{4\pi r^3}\mathbf{n}(\hat{z}\cdot\mathbf{n})\right] \tag{6.20-6}$$

where the factor of 2/3 multiplying the delta function comes from treating the gradient of z/r^3 according to (4.20).

Solution: The general formula for the transverse current is

$$\begin{aligned}\mathbf{J}_T = \mathbf{J} - \epsilon_0\nabla\dot{\Phi} &= -\delta^3(\mathbf{r})\delta'(t)\hat{z} - \epsilon_0\nabla\left[-\frac{1}{4\pi\epsilon_0}\delta'(t)\frac{z}{r^3}\right] \\ &= -\delta^3(\mathbf{r})\delta'(t)\hat{z} + \frac{1}{4\pi}\delta'(t)\nabla\left[\frac{z}{r^3}\right].\end{aligned} \tag{6.20-7}$$

There is an easy way and a hard way to go about this. I could identify the gradient term as the electric field of a dipole centered at the origin with $\mathbf{p} = \hat{z}$, but I will prove that fact directly through the explicit computation of the gradient:

$$\begin{aligned}\nabla\left[\frac{z}{r^3}\right] = \nabla\left[\frac{\cos\theta}{r^2}\right] = \nabla\left[\hat{z}\cdot\frac{\hat{r}}{r^2}\right] &= -\nabla\left[\hat{z}\cdot\nabla\left(\frac{1}{r}\right)\right] \\ &= -\hat{e}_i\frac{\partial}{\partial x^i}\frac{\partial}{\partial z}\left(\frac{1}{r}\right) \\ &= -\hat{e}_i\left[\frac{3x^iz}{r^5} - \frac{\delta^{i3}}{r^3} - \frac{4\pi}{3}\delta^{i3}\delta^3(\mathbf{r})\right] \\ &= \frac{4\pi}{3}\delta^3(\mathbf{r})\hat{z} + \frac{\hat{z}}{r^3} - \frac{3\hat{r}(\hat{z}\cdot\hat{r})}{r^3},\end{aligned} \tag{6.20-8}$$

where I used Woodard's hint and noted $x^i\hat{e}_i = \mathbf{r}$, $\hat{z}\cdot\mathbf{r} = r\cos\theta$. Plugging this into (2.7),

$$\mathbf{J}_T = -\delta'(t)\left[\delta^3(\mathbf{r})\hat{z} - \frac{1}{3}\delta^3(\mathbf{r})\hat{z} - \frac{\hat{z}}{4\pi r^3} + \frac{3\hat{r}(\hat{z}\cdot\hat{r})}{4\pi r^3}\right], \tag{6.20-9}$$

which is simply

$$\boxed{\mathbf{J}_T(\mathbf{r},t) = -\delta'(t)\left[\frac{2}{3}\delta^3(\mathbf{r})\hat{z} - \frac{\hat{z}}{4\pi r^3} + \frac{3\hat{r}(\hat{z}\cdot\hat{r})}{4\pi r^3}\right]} \tag{6.20-10}$$

as desired (6.20-6) with the identification $\mathbf{n} = \hat{r}$.

(c) Show that the electric and magnetic fields are causal and that the electric field components are

$$E_x(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\frac{c}{r}\left[-\delta''(r-ct) + \frac{3}{r}\delta'(r-ct) - \frac{3}{r^2}\delta(r-ct)\right]\sin\theta\cos\theta\cos\phi, \tag{6.20-11}$$

E_y is the same as E_x, with $\cos\phi \to \sin\phi$, and

$$E_z(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\frac{c}{r}\left[\sin^2\theta\,\delta''(r-ct) + (3\cos^2\theta - 1)\left(\frac{\delta'(r-ct)}{r} - \frac{\delta(r-ct)}{r^2}\right)\right] \tag{6.20-12}$$

<u>Hint:</u> While the answer in part b displays the transverse current explicitly, the less explicit form

$$\mathbf{J}_t(\mathbf{x},t) = -\delta'(t)\left[\hat{z}\delta(\mathbf{x}) + \frac{1}{4\pi}\nabla\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right] \tag{6.20-13}$$

can be used with (6.47) to calculate the vector potential and the fields for part c. An alternative method is to use the Fourier transforms in time of $\mathbf{J}_t$ and $\mathbf{A}$, the Green's function (6.40) and its spherical wave expansion from Chapter 9.

Solution: The instantaneous Coulomb potential contributes to the electric field $\mathbf{E} = -\nabla\Phi - \frac{1}{c}\dot{\mathbf{A}}$ in the following way,

$$-\nabla\Phi = \frac{1}{4\pi\epsilon_0}\delta(t)\nabla\left[\frac{z}{r^3}\right]. \tag{6.20-14}$$

The vector potential requires more work

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int d^3r'\frac{[\mathbf{J}(\mathbf{r}',t')]_{\text{ret}}}{||\mathbf{r}-\mathbf{r}'||} = -\frac{\mu_0}{4\pi}\int d^3r'\delta'\left(t-\frac{1}{c}||\mathbf{r}-\mathbf{r}'||\right)\frac{\hat{z}\delta^3(\mathbf{r})+\frac{1}{4\pi}\frac{\partial}{\partial z'}\nabla'\left[\frac{1}{r'}\right]}{||\mathbf{r}-\mathbf{r}'||} \\ &= -\frac{\mu_0}{4\pi r}\delta'\left(t-\frac{r}{c}\right)\hat{z} - \frac{\mu_0}{(4\pi)^2}\int d^3r'\delta'\left(t-\frac{1}{c}||\mathbf{r}-\mathbf{r}'||\right)\frac{\frac{\partial}{\partial z'}\nabla'\left[\frac{1}{r'}\right]}{||\mathbf{r}-\mathbf{r}'||}\end{aligned} \tag{6.20-15}$$

Following Jackson's notation, I will let $\mathbf{R} = \mathbf{r}-\mathbf{r}'$, such that $r' = ||\mathbf{R}-\mathbf{r}||$ and $d^3r' \to d^3R$:

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= -\frac{\mu_0}{4\pi r}\delta'\left(t-\frac{r}{c}\right)\hat{z} - \frac{\mu_0}{(4\pi)^2}\int d^3R\delta'\left(t-\frac{R}{c}\right)\frac{1}{R}\frac{\partial}{\partial z_R}\nabla_R\left[\frac{1}{||\mathbf{R}-\mathbf{r}||}\right] \\ &= -\frac{\mu_0}{4\pi r}\delta'\left(t-\frac{r}{c}\right)\hat{z} - \frac{\mu_0}{(4\pi)^2}\frac{\partial}{\partial z}\nabla\int d^3R\,\delta'\left(t-\frac{R}{c}\right)\frac{1}{R}\left[\frac{1}{||\mathbf{R}-\mathbf{r}||}\right],\end{aligned} \tag{6.20-16}$$

where I noted that $\frac{\partial}{\partial z_R}\nabla_R(||\mathbf{R}-\mathbf{r}||) = (-1)^2\frac{\partial}{\partial z}\nabla\left(||\mathbf{R}-\mathbf{r}||\right)$. I am now in the position to make arguments about the presence of the delta function. Expanding the integrand,

$$\mathbf{A}(\mathbf{r},t) = -\frac{\mu_0}{4\pi r}\delta'\left(t-\frac{r}{c}\right)\hat{z} - \frac{\mu_0}{(4\pi)^2}\frac{\partial}{\partial z}\nabla\int d^3R\,\delta'\left(t-\frac{R}{c}\right)\frac{1}{R}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{4\pi}{2l+1}\sum_{m=-\infty}^{\infty}Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi'), \tag{6.20-17}$$

where $r_{<,>} \in \{R, r\}$, it is evident that only the $l=m=0$ term will contribute. This is because I have the quantity

$$\int d\Omega'\,Y_{lm}^*(\theta',\phi') = \sqrt{4\pi}\delta_{l,0}\delta_{m,0} \tag{6.20-18}$$

which clearly enforces this condition. I find,

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= -\frac{\mu_0}{4\pi r}\delta'\left(t-\frac{r}{c}\right)\hat{z} - \frac{\mu_0}{4\pi}\frac{\partial}{\partial z}\nabla\int dR\,\delta'\left(t-\frac{R}{c}\right)\frac{R}{r_>} \\ &= -\frac{\mu_0}{4\pi r}\delta'\left(t-\frac{r}{c}\right)\hat{z} - \frac{\mu_0}{4\pi}\frac{\partial}{\partial z}\nabla\left[\frac{1}{r}\int_0^r dR\,R\delta'\left(t-\frac{R}{c}\right) + \int_r^\infty dR\,\delta'\left(t-\frac{R}{c}\right)\right].\end{aligned} \tag{6.20-19}$$

Before I perform the integrations, I will first differentiate with respect to time, using the fact that $\delta(t-R/c) = c\delta(ct-R) \implies \delta'(t-R/c) = c^2\delta'(ct-R) \implies \delta''(t-R/c) = c^3\delta''(ct-R)$:

$$\begin{aligned}-\dot{\mathbf{A}}(\mathbf{r},t) &= \frac{\mu_0 c^3}{4\pi r}\delta''(ct-r)\,\hat{z} + \frac{\mu_0 c^3}{4\pi}\frac{\partial}{\partial z}\nabla\left[\frac{1}{r}\int_0^r dR\,R\delta''(ct-R) + \int_r^\infty dR\,\delta''(ct-R)\right] \\ &= \frac{c}{4\pi\epsilon_0 r}\delta''(ct-r)\,\hat{z} + \frac{c}{4\pi\epsilon_0}\frac{\partial}{\partial z}\nabla\left[\frac{1}{r}\left(R\delta'(ct-R)\Big|_0^r - \int_0^r dR\,\delta'(ct-R)\right) + \delta'(ct-R)\Big|_r^\infty\right] \\ &= \frac{c}{4\pi\epsilon_0 r}\delta''(ct-r)\,\hat{z} + \frac{c}{4\pi\epsilon_0}\frac{\partial}{\partial z}\nabla\left[\frac{1}{r}\left(\cancel{r\delta'(ct-r)} - \delta(ct-r) + \delta(ct)\right) - \cancel{\delta'(ct-r)}\right] \\ &= \frac{c}{4\pi\epsilon_0 r}\delta''(ct-r)\,\hat{z} + \frac{c}{4\pi\epsilon_0}\frac{\partial}{\partial z}\nabla\left[\frac{1}{r}\left(-\delta(ct-r) + \delta(ct)\right)\right],\end{aligned} \tag{6.20-20}$$

$$\begin{aligned}\therefore -\dot{\mathbf{A}}(\mathbf{r},t) &= \frac{c}{4\pi\epsilon_0 r}\delta''(r-ct)\,\hat{z} + \frac{c}{4\pi\epsilon_0}\frac{\partial}{\partial z}\nabla\left[\frac{1}{r}\left(\delta(ct) - \delta(r-ct)\right)\right] \\ &= \frac{c}{4\pi\epsilon_0 r}\delta''(r-ct)\,\hat{z} + \frac{c}{4\pi\epsilon_0}\left[-\nabla\left[\frac{z}{r^3}\right]\left(\delta(ct)-\delta(r-ct)\right) + \frac{2z\delta'(r-ct)}{r^3}\hat{r} - \frac{1}{r}\nabla\left[\delta'(r-ct)\frac{z}{r}\right]\right] \\ &= \frac{c}{4\pi\epsilon_0 r}\delta''(r-ct)\,\hat{z} + \frac{c}{4\pi\epsilon_0}\left[\nabla\left[\frac{z}{r^3}\right]\left(\delta(r-ct)-\delta(ct)\right) - \delta''(r-ct)\frac{z}{r^2}\hat{r}\right. \\ &\qquad \left. + \frac{2z\delta'(r-ct)}{r^3}\hat{r} + \delta'(r-ct)\frac{\sin\theta}{r^2}\hat{\theta}\right]\end{aligned} \tag{6.20-21}$$

Using (6.20-8) for the remaining gradient,

$$
\begin{aligned}
-\dot{\mathbf{A}}(\mathbf{r},t) = \frac{c}{4\pi\epsilon_0 r}\delta''\left(r-ct\right)\hat{z} - \frac{1}{4\pi\epsilon_0}\delta(t)\nabla\left[\frac{z}{r^3}\right] + \frac{c}{4\pi\epsilon_0}\Bigg[& \left(\frac{\hat{z}}{r^3} - \frac{3\hat{r}(\hat{z}\cdot\hat{r})}{r^3}\right)\delta\left(r-ct\right) \\
& - \delta''(r-ct)\frac{z}{r^2}\hat{r} + \frac{2\cos\theta\delta'(r-ct)}{r^2}\hat{r} + \delta'(r-ct)\frac{\sin\theta}{r^2}\hat{\theta}\Bigg]
\end{aligned}
\tag{6.20-22}
$$

Using $\sin\theta\hat{\theta} = \cos\theta\hat{r} - \hat{z}$, and moving the now generated $\nabla\Phi$ to the other side,

$$
\begin{aligned}
-\nabla\Phi - \dot{\mathbf{A}}(\mathbf{r},t) &= \frac{c}{4\pi\epsilon_0 r}\delta''\left(r-ct\right)\hat{z} + \frac{c}{4\pi\epsilon_0}\Bigg[\left(\frac{\hat{z}}{r^3} - \frac{3\hat{r}\cos\theta}{r^3}\right)\delta\left(r-ct\right) \\
&\quad - \delta''(r-ct)\frac{z}{r^2}\hat{r} + \delta'(r-ct)\frac{3\cos\theta}{r^2}\hat{r} - \delta'(r-ct)\frac{\hat{z}}{r^2}\Bigg] \\
&= \frac{c}{4\pi\epsilon_0 r}\left[\delta''(r-ct)\left(\hat{z}-\cos\theta\hat{r}\right) + \frac{\delta'(r-ct)}{r}\left(3\cos\theta\hat{r}-\hat{z}\right) - \frac{\delta\left(r-ct\right)}{r^2}\left(3\cos\theta\hat{r}-\hat{z}\right)\right]
\end{aligned}
\tag{6.20-23}
$$

Now, note that since

$$
\begin{aligned}
\left[\hat{z} - a\cos\theta\hat{r}\right]_x &= -a\sin\theta\cos\theta\cos\phi \\
\left[\hat{z} - a\cos\theta\hat{r}\right]_y &= -a\sin\theta\cos\theta\sin\phi \\
\left[\hat{z} - a\cos\theta\hat{r}\right]_z &= 1 - a\cos^2\theta
\end{aligned}
\tag{6.20-24}
$$

where in our problem $a = 1, 3$, I find from (6.20-23) the components of

$$
\boxed{E_x(\mathbf{r},t) = \frac{c}{4\pi\epsilon_0 r}\left[-\delta''(r-ct) + \frac{3\delta'(r-ct)}{r} - \frac{3\delta(r-ct)}{r^2}\right]\sin\theta\cos\theta\cos\phi}
\tag{6.20-25}
$$

$$
\boxed{E_y(\mathbf{r},t) = \frac{c}{4\pi\epsilon_0 r}\left[-\delta''(r-ct) + \frac{3\delta'(r-ct)}{r} - \frac{3\delta(r-ct)}{r^2}\right]\sin\theta\cos\theta\sin\phi}
\tag{6.20-26}
$$

$$
\boxed{E_z(\mathbf{x},t) = \frac{c}{4\pi\epsilon_0 r}\left[\sin^2\theta\delta''(r-ct) + (3\cos^2\theta - 1)\left(\frac{\delta'(r-ct)}{r} - \frac{\delta(r-ct)}{r^2}\right)\right]}
\tag{6.20-27}
$$

as desired.

Chapter 7

Jackson (7.3)

Two plane semi-infinite slabs of the same uniform, isotropic, non-permeable, lossless dielectric with index of refraction n are parallel and separated by an air gap ($n = 1$) of width d. A plane electromagnetic wave of frequency ω is incident on the gap from one of the slabs with angle of incidence i. For linear polarization both parallel to and perpendicular to the plane of incidence,

(a) calculate the ratio of power transmitted into the second slab to the incident power and ratio of reflected to incident power;

Solution: Inspect the following figure, Here $n = \sin i = \sin\theta$, $\mathbf{B}_0 = \frac{n}{c}\hat{z} \times \mathbf{E}_0$. Woodard has provided us with the

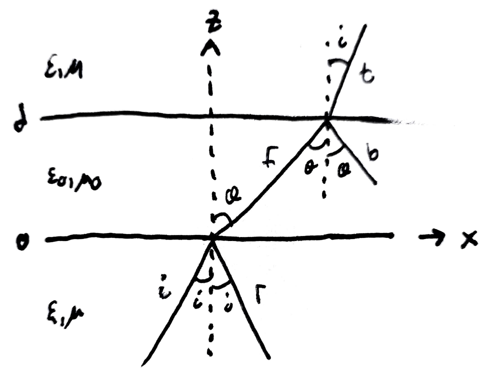


Figure 8: Jackson (7.3)

electric and magnetic fields in each region, which can be easily derived from $\mathbf{k}$ in each region:

$$\begin{aligned}
E_{0,i} &= \alpha_i \hat{y} + \beta_i(\cos i\hat{x} - \sin i\hat{z}) \\
E_{0,r} &= \alpha_r \hat{y} + \beta_r(\cos i\hat{x} + \sin i\hat{z}) \\
E_{0,f} &= \alpha_f \hat{y} + \beta_f(\cos \theta\hat{x} - \sin \theta\hat{z}) \\
E_{0,b} &= \alpha_b \hat{y} + \beta_b(\cos \theta\hat{x} + \sin \theta\hat{z}) \\
E_{0,t} &= \alpha_t \hat{y} + \beta_t(\cos i\hat{x} - \sin i\hat{z})
\end{aligned} \tag{7.3-1}$$

$$\begin{aligned}
\frac{c}{n}B_{0,i} &= \alpha_i(-\cos i\hat{x} + \sin i\hat{z}) + \beta_i\hat{y} \\
\frac{c}{n}B_{0,r} &= \alpha_r(\cos i\hat{x} + \sin i\hat{z}) - \beta_r\hat{y} \\
\frac{c}{n}B_{0,f} &= \alpha_i(-\cos \theta\hat{x} + \sin \theta\hat{z}) + \beta_f\hat{y} \\
\frac{c}{n}B_{0,b} &= \alpha_b(\cos \theta\hat{x} + \sin \theta\hat{z}) - \beta_b\hat{y} \\
\frac{c}{n}B_{0,t} &= \alpha_t(-\cos i\hat{x} + \sin i\hat{z}) + \beta_t\hat{y}
\end{aligned} \tag{7.3-2}$$

Everything will be written in terms of a given α_i, β_i. There are two boundary conditions at each of the two boundaries, $\Delta\mathbf{E}_{\|} = 0$ and $\Delta\mathbf{H}_{\|} = 0$, giving rise to 4 equations. Each of the 4 equations will be a mixture of two directions, giving rise to an effective 8 equations in 8 unknowns. The boundary condition at d requires the addition of phase factors ϕ_θ and ϕ_t on the fields due to their evaluation at $z = 0$. I will omit the phase on the transmitted field because it can be shown to provide an overall phase that is unobservable after taking the absolute value squared (to get the power ratio). In addition, Woodard has us taking $\mu = \mu_0$, so the boundary condition $\Delta\mathbf{H}_{\|} = \Delta\mathbf{B}_{\|} = 0$. Since $\hat{n} = \hat{z}$ then each boundary condition involves curling it with $\hat{z}$, resulting in two equations, one in $\hat{x}$ and the other in $\hat{y}$. The conditions $\Delta\mathbf{E}_{\|}\Big|_{z=0} = 0$,

$\Delta\mathbf{B}_\parallel\Big|_{z=0} = 0$, $\Delta\mathbf{E}_\parallel\Big|_{z=d} = 0$, and $\Delta\mathbf{B}_\parallel\Big|_{z=d} = 0$ give,

$$\begin{aligned}\alpha_i + \alpha_r &= \alpha_f + \alpha_b \\ \alpha_f e^{i\phi_\theta} + \alpha_b e^{-i\phi_\theta} &= \alpha_t \\ n(\alpha_r - \alpha_i)\cos i &= (\alpha_b - \alpha_f)\cos\theta \\ (-\alpha_f e^{i\phi_\theta} + \alpha_b e^{-i\phi_\theta})\cos\theta &= -n\alpha_t \cos i\end{aligned} \tag{7.3-3}$$

for α's and

$$\begin{aligned}(\beta_i + \beta_r)\cos i &= (\beta_f + \beta_b)\cos\theta \\ (\beta_f e^{i\phi_\theta} + \beta_b e^{-i\phi_\theta})\cos\theta &= \beta_t \cos i \\ n(-\beta_i + \beta_r) &= -\beta_f + \beta_t \\ -\beta_f e^{i\phi_\theta} + \beta_b e^{-i\phi_\theta} &= -n\beta_t,\end{aligned} \tag{7.3-4}$$

where note that the acquired phase is $\phi_\theta = \frac{\omega d}{c}\cos\theta$, and the reflected rays get $e^{-i\phi_\theta}$ instead of the positive sign. I will first work with the α's, so first define $\gamma = \frac{n\cos i}{\cos\theta} = \frac{n\cos i}{\sqrt{1-\sin^2 i}}$. With this choice, equations (1) and (3) of (2.3) can be recast into

$$\begin{aligned}2\alpha_f &= \alpha_i(1+\gamma) + \alpha_r(1-\gamma) \\ 2\alpha_b &= \alpha_i(1-\gamma) + \alpha_r(1+\gamma),\end{aligned} \tag{7.3-5}$$

and equations (2) and (4) of (7.3-3) are simply

$$\begin{aligned}2\alpha_f &= \alpha_t(1+\gamma)e^{-i\phi_\theta} \\ 2\alpha_b &= \alpha_t(1-\gamma)e^{i\phi_\theta},\end{aligned} \tag{7.3-6}$$

where clearly (7.3-5) and (7.3-6) can be equated to result in

$$\alpha_r\frac{1+\gamma}{1-\gamma} = \alpha_t e^{-i\phi_\theta} - \alpha_i \qquad\qquad \alpha_r\frac{1-\gamma}{1+\gamma} = \alpha_t e^{i\phi_\theta} - \alpha_i. \tag{7.3-7}$$

My goal is to first find α_t/α_i, which is the probability density of transmission in the perpendicular polarization case. Dividing the two equations (7.3-7) results in

$$\begin{aligned}\frac{(1-\gamma)^2}{(1+\gamma)^2} = \frac{\frac{\alpha_t}{\alpha_i}e^{-i\phi_\theta} - 1}{\frac{\alpha_t}{\alpha_i}e^{i\phi_\theta} - 1} \iff 1 - \frac{(1-\gamma)^2}{(1+\gamma)^2} &= \frac{\alpha_t}{\alpha_i}\left[e^{-i\phi_\theta} - \frac{(1-\gamma)^2}{(1+\gamma)^2}e^{i\phi_\theta}\right] \\ \frac{\alpha_t}{\alpha_i} &= \frac{4\gamma}{(1+\gamma)^2 e^{-i\phi_\theta} - (1-\gamma)^2 e^{i\phi_\theta}}.\end{aligned} \tag{7.3-8}$$

The norm squared of this quantity is the ratio of power transmitted to incident power,

$$\begin{aligned}T_\perp = \left|\frac{\alpha_t}{\alpha_i}\right|^2 &= \frac{16\gamma^2}{[(1+\gamma)^2 e^{-i\phi_\theta} - (1-\gamma)^2 e^{i\phi_\theta}][(1+\gamma)^2 e^{i\phi_\theta} - (1-\gamma)^2 e^{-i\phi_\theta}]} \\ &= \frac{16\gamma^2}{(1+\gamma)^4 - 2(1+\gamma)^2(1-\gamma)^2\cos(2\phi_\theta) + (1-\gamma)^4} \\ &= \frac{16\gamma^2}{16\gamma^2 + 4(1-\gamma^2)^2\sin^2\phi_\theta},\end{aligned} \tag{7.3-9}$$

$$\boxed{T_\perp = \frac{4\gamma^2}{4\gamma^2 + (1-\gamma^2)^2\sin^2\phi_\theta}} \tag{7.3-10}$$

This is ratio of power transmitted to the incident power for the case of perpendicular polarization. With this, the computation of the ratio of power reflected to incident power is simple as $1 = T_\perp + R_\perp$ implies from (7.3-10) that

$$\boxed{R_\perp = \frac{(1-\gamma^2)^2 \sin^2 \phi_\theta}{4\gamma^2 + (1-\gamma^2)^2 \sin^2 \phi_\theta}} \tag{7.3-11}$$

is the ratio of power reflected to incident power in the case of perpendicular polarization.

The equivalent quantities for parallel polarization are determined by the β's. Define now that quantity $\eta = \frac{\cos i}{n \cos \theta} = \frac{\cos i}{n\sqrt{1-n^2 \sin^2 i}}$, such that $\gamma = n^2 \eta$. With this, equations (1) and (3) of (2.4) imply

$$\begin{aligned} 2\beta_f &= n\beta_i(\eta+1) + n\beta_r(\eta-1) \\ 2\beta_b &= n\beta_i(\eta-1) + n\beta_r(\eta+1) \end{aligned} \tag{7.3-12}$$

and equations (2) and (4) of (2.4) imply

$$\begin{aligned} 2\beta_f &= n\beta_t(\eta+1)e^{-i\phi_\theta} \\ 2\beta_b &= n\beta_t(\eta-1)e^{i\phi_\theta}, \end{aligned} \tag{7.3-13}$$

which all together result in the equations

$$\beta_r \frac{\eta+1}{\eta-1} = \beta_t e^{i\phi_\theta} - \beta_i \qquad\qquad \beta_r \frac{\eta-1}{\eta+1} = \beta_t e^{-i\phi_\theta} - \beta_i \tag{7.3-14}$$

that can be divided to give

$$\frac{(1-\eta)^2}{(1+\eta)^2} = \frac{\frac{\beta_t}{\beta_i}e^{-i\phi_\theta} - 1}{\frac{\beta_t}{\beta_i}e^{i\phi_\theta} - 1}. \tag{7.3-15}$$

This is great, because this equation is the same as that for the case of perpendicular polarization (7.3-8) but with the substitutions $\alpha \to \beta$ and $\gamma \to \eta$. I can immediately quote the same answer (7.3-10) under these substitutions,

$$\boxed{T_\parallel = \frac{4\eta^2}{4\eta^2 + (1-\eta)^2 \sin^2 \phi_0} = \frac{4\gamma^2}{4\gamma^2 + \left(n^2 - \frac{\gamma^2}{n^2}\right)^2 \sin^2 \phi_\theta}} \tag{7.3-16}$$

where I recalled that $\gamma = n^2\eta$ so I can write both transmission probabilities in terms of the same parameter γ. This is the ratio of transmitted power to incident power in the case of parallel polarization. Since $1 = T_\parallel + R_\parallel$, it follows that

$$\boxed{R_\parallel = \frac{\left(n^2 - \frac{\gamma^2}{n^2}\right)^2 \sin^2 \phi_\theta}{4\gamma^2 + \left(n^2 - \frac{\gamma^2}{n^2}\right)^2 \sin^2 \phi_\theta}} \tag{7.3-17}$$

is the ratio of reflected power to incident power in the case of parallel polarization.

(b) for i greater than the critical angle for total internal reflection, sketch the ratio of transmitted power to incident power as a function of d measured in units of wavelength in the gap.

Solution: Total internal reflection occurs when $\sqrt{1-n^2\sin^2 i}$ becomes complex, implying that $\sin i > 1/n$ is the condition for T.I.R. Substituting the necessary quantities:

$$\begin{aligned} T_\perp &= \frac{4n^2\cos^2 i}{4n^2\cos^2 i - |1-n^2\sin^2 i|\left(1 + \frac{n^2\cos^2 i}{|1-n^2\sin^2 i|}\right)^2 \sin^2\left(\frac{i\omega d}{c}\sqrt{n^2\sin^2 i - 1}\right)} \\ &= \frac{4n^2\cos^2 i}{4n^2\cos^2 i + |1-n^2\sin^2 i|\left(1 + \frac{n^2\cos^2 i}{|1-n^2\sin^2 i|}\right)^2 \sinh^2\left(\frac{\omega d}{c}\sqrt{n^2\sin^2 i - 1}\right)} \\ \implies T_\parallel &= \frac{4n^2\cos^2 i}{4n^2\cos^2 i + |1-n^2\sin^2 i|\left(n^2 + \frac{\cos^2 i}{|1-n^2\sin^2 i|}\right)^2 \sinh^2\left(\frac{\omega d}{c}\sqrt{n^2\sin^2 i - 1}\right)}, \end{aligned} \tag{7.3-18}$$

where I noted that $\sin^2(ix) = -\sinh^2(x)$. For the purposes of plotting, I will measure d in units of c/ω such that I can let $c/\omega = 1$, and I will select $n = 3/2$, $\sin i = \sqrt{3}/2$ (for a $\pi/3$ angle of incidence). I then find the following figure

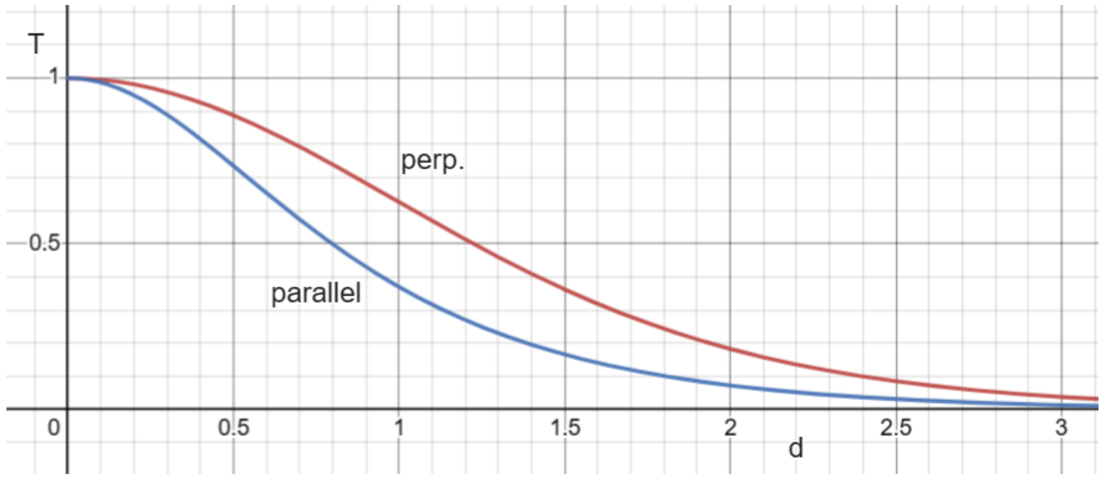


Figure 9: Plot of T vs. d in units of gap wavelength for $\perp$ polarization (red) and $\parallel$ polarization (blue) for $n = 3/2$ and $i = \pi/3 = 60°$. There is a critical gap $d_c \approx 1 = c/\omega$ for which the greatest difference in transmission probability occurs between $\perp$ and $\parallel$ polarization.

Jackson (7.16)

Plane waves propagate in a homogeneous, non-permeable, but anisotropic dielectric. The dielectric is characterized by a tensor ε_{ij}, but if coordinate axes are chosen as the principle axes, the components of displacement along these axes are related to the electric-field components by $D_i = \varepsilon_i E_i$, where ε_i are the eigenvalues of the matrix of ε_{ij}.

(a) Show that the plane waves with frequency ω and wave vector $\mathbf{k}$ must satisfy

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \mu_0 \omega^2 \mathbf{D} = 0. \tag{7.16-1}$$

Solution: Faraday's Law tells me that

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \implies i\mathbf{k} \times \mathbf{E}_0 - i\omega \mathbf{B}_0 = 0, \tag{7.16-2}$$

where I assumed the fields are plane waves with $\exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t)$. This dielectric leaves unaffected the magnetic field, as there is no magnetization, meaning $\mathbf{B} = \mu_0 \mathbf{H}$ and

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0) - \mu_0 \omega \mathbf{k} \times \mathbf{H}_0 = 0, \tag{7.16-3}$$

where I additionally divided by i and hit both sides with $\mathbf{k}$. Next, the Ampere-Maxwell law tells me that

$$\nabla \times \mathbf{H} - \partial_t \mathbf{D} = 0 \implies \mathbf{k} \times \mathbf{H}_0 + \omega \mathbf{D}_0 = 0, \tag{7.16-4}$$

where I again assumed plane wave solutions and divided by i. Substituting this into (7.16-3), I find

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0) + \mu_0 \omega^2 \mathbf{D}_0 = 0 \implies \boxed{\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \mu_0 \omega^2 \mathbf{D} = 0} \tag{7.16-5}$$

as desired. Note that I multiplied by the exponential factor to convert all fields into their full forms.

(b) Show that for a given wave vector $\mathbf{k} = k\hat{n}$ there are two distinct modes of propagation with different phase velocities $v = \omega/k$ that satisfy the Fresnel equation

$$\sum_{i=1}^{3} \frac{n_i^2}{v^2 - v_i^2} = 0$$

where $v_i = 1/\sqrt{\mu_0 \epsilon_i}$ is called a principal velocity, and n_i is the component of $\hat{n}$ along the i-th principal axis.

Solution: Expand the vector triple product,

$$\mathbf{k}(\mathbf{k} \cdot \mathbf{E}) - \mathbf{E}(\mathbf{k} \cdot \mathbf{k}) + \mu_0 \omega^2 \mathbf{D} = 0, \tag{7.16-6}$$

where since $\mathbf{k} \cdot \mathbf{k} = k^2 = \omega^2/v^2$ such that

$$\begin{aligned} 0 = \frac{\omega^2}{v^2}\hat{n}(\hat{n} \cdot \mathbf{E}) - \frac{\omega^2}{v^2}\mathbf{E} + \mu_0 \omega^2 \mathbf{D} &= \hat{n}(\hat{n} \cdot \mathbf{E}) - \mathbf{E} + \mu_0 v^2 \mathbf{D} \\ &= (n_i n_j - \delta_{ij})E_j + \mu_0 v^2 \epsilon_i E_i \\ &= \left[(n_i n_j - \delta_{ij}) + \frac{\delta_{ij} v^2}{v_j^2}\right] E_j, \end{aligned} \tag{7.16-7}$$

which is a matrix equation that gives non-zero solutions if the determinant of the matrix in brackets vanishes,

$$\begin{aligned} 0 &= \begin{vmatrix} n_1^2 - 1 + \frac{v^2}{v_1^2} & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2^2 - 1 + \frac{v^2}{v_2^2} & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3^2 - 1 + \frac{v^2}{v_3^2} \end{vmatrix} \\ &= \left(n_1^2 + \frac{v^2 - v_1^2}{v_1^2}\right)\left[\left(n_2^2 + \frac{v^2 - v_2^2}{v_2^2}\right)\left(n_3^2 + \frac{v^2 - v_3^2}{v_3^2}\right) - n_2^2 n_3^2\right] + n_1^2 n_2^2 \left(\frac{v^2 - v_3^2}{v_3^2}\right) + n_1^2 n_3^2 \left(\frac{v^2 - v_2^2}{v_2^2}\right) \\ &= \frac{v^2}{v_1^2 v_2^2 v_3^3}\left[n_1^2(v^2 - v_2^2)(v^2 - v_3^2) + n_2^2(v^2 - v_1^2)(v^2 - v_3^2) + n_3^2(v^2 - v_1^2)(v^2 - v_2^2)\right], \end{aligned} \tag{7.16-8}$$

where I suppressed the algebra for obvious reasons. Which for the nontrivial solution $v \neq 0$, the quantity in brackets vanishes,

$$\begin{aligned} 0 &= n_1^2(v^2 - v_2^2)(v^2 - v_3^2) + n_2^2(v^2 - v_1^2)(v^2 - v_3^2) + n_3^2(v^2 - v_1^2)(v^2 - v_2^2) \\ &= \frac{n_1^2}{v^2 - v_1^2} + \frac{n_2^2}{v^2 - v_2^2} + \frac{n_3^2}{v^2 - v_3^2} \implies \boxed{\sum_{i=1}^{3} \frac{n_i^2}{v^2 - v_i^2} = 0} \end{aligned} \tag{7.16-9}$$

which is the Fresnel equation.

(c) Show that $\mathbf{D}_a \cdot \mathbf{D}_b = 0$, where $\mathbf{D}_a$ and $\mathbf{D}_b$ are the displacements associated with the two modes of propagation.

Solution: Take (7.16-7) for two modes a and b,

$$0 = \left[(n_i n_j - \delta_{ij}) + \mu_0 v_a^2 \epsilon_i \delta_{ij}\right] E_j^a, \qquad 0 = \left[(n_i n_j - \delta_{ij}) + \mu_0 v_b^2 \epsilon_i \delta_{ij}\right] E_j^b, \tag{7.16-10}$$

Act each equation with the mode of the other equation,

$$0 = \left[(n_i n_j - \delta_{ij}) + \mu_0 v_a^2 \epsilon_i \delta_{ij}\right] E_j^a E_i^b, \qquad 0 = \left[(n_i n_j - \delta_{ij}) + \mu_0 v_b^2 \epsilon_i \delta_{ij}\right] E_i^a E_j^b, \tag{7.16-11}$$

where since the tensor in brackets is symmetric, I am free to swap $i \leftrightarrow j$ in say, the first equation:

$$0 = \left[(n_i n_j - \delta_{ij}) + \mu_0 v_a^2 \epsilon_i \delta_{ij}\right] E_i^a E_j^b, \qquad 0 = \left[(n_i n_j - \delta_{ij}) + \mu_0 v_b^2 \epsilon_i \delta_{ij}\right] E_i^a E_j^b. \tag{7.16-12}$$

These equations together imply that

$$0 = \mu_0(v_a^2 - v_b^2)\epsilon_i \delta_{ij} E_i^a E_j^b = (v_a^2 - v_b^2)\mathbf{D}^a \cdot \mathbf{E}^b \implies \mathbf{D}_a \cdot \mathbf{E}_b = 0, \tag{7.16-13}$$

for any two arbitrary modes. I will now take (7.16-7) in its vector form for some mode a and dot it with some other $\mathbf{D}_b$,

$$0 = (\hat{n} \cdot \mathbf{D}_b)(\hat{n} \cdot \mathbf{E}_a) - \mathbf{E}_a \cdot \mathbf{D}_b + \mu_0 v_a^2 \mathbf{D}_a \cdot \mathbf{D}_b \iff \mu_0 v_a^2 \mathbf{D}_a \cdot \mathbf{D}_b = \mathbf{E}_a \cdot \mathbf{D}_b, \tag{7.16-14}$$

which followed from Gauss' Law that $\hat{n} \cdot \mathbf{E} = 0$. Thus,

$$\boxed{\mathbf{D}_a \cdot \mathbf{D}_b = 0} \tag{7.16-15}$$

as desired.

Jackson (7.27)

The angular momentum of a distribution of electromagnetic fields in vacuum is given by

$$\mathbf{L} = \frac{1}{\mu_0 c^2} \int d^3 r \, \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \tag{7.27-1}$$

where the integration is over all space.

(a) For fields produced a finite time in the past (and so localized to a finite region of space) show that, provided the magnetic field is eliminated in favor of the vector potential $\mathbf{A}$, the angular momentum can be written in the form

$$\mathbf{L} = \frac{1}{\mu_0 c^2} \int d^3 r \left[\mathbf{E} \times \mathbf{A} + \sum_{j=1}^{3} E_j (\mathbf{r} \times \nabla) A_j \right]. \tag{7.27-2}$$

The first term is sometimes identified with the "spin" of the photon and the second with its "orbital" angular momentum because of the presence of the angular momentum operator $\mathbf{L}_{\text{op}} = -i(\mathbf{r} \times \nabla)$.

Solution: First, I will decompose the double vector product

$$\begin{aligned} \mathbf{E} \times (\nabla \times \mathbf{A}) &= \mathbf{E} \times \varepsilon^{ijk} \partial_j A_k e_i \\ &= \varepsilon^{ijk} \varepsilon^{iml} E_l \partial_j A_k e_m \\ &= \delta_{jm} \delta_{kl} E_l \partial_j A_k e_m - \delta_{jl} \delta_{km} E_l \partial_j A_k e_m \\ &= E_k \partial_j A_k e_j - E_j \partial_j A_k e_k \end{aligned} \tag{7.27-3}$$

where e_j is a unit vector in direction j. Therefore, I find from (7.27-1) that

$$\begin{aligned} \mathbf{L} &= \frac{1}{\mu_0 c^2} \int d^3 r \, E_k (\mathbf{r} \times \nabla) A_k - \frac{1}{\mu_0 c^2} \epsilon^{ilk} e_i \int d^3 r \, r_l E_j \partial_j A_k \\ &= \frac{1}{\mu_0 c^2} \int d^3 r \, E_k (\mathbf{r} \times \nabla) A_k - \frac{1}{\mu_0 c^2} \epsilon^{ilk} e_i \int d^3 r \, \left[\partial_j (r_l E_k A_k) - (\partial_j r_l) E_j A_k - r_l (\partial_j E_j) A_k \right], \end{aligned} \tag{7.27-4}$$

where I expanded the second integral as a total derivative minus 2 terms. This is nice, because the total derivative vanishes for localized sources, and since we are in vacuum, $\partial_j E_j = \rho = 0$, so using the fact that $\partial_j r_l = \delta_{jl}$ I find that

$$\begin{aligned} \mathbf{L} &= \frac{1}{\mu_0 c^2} \int d^3 r \, E_k (\mathbf{r} \times \nabla) A_k - \frac{1}{\mu_0 c^2} \epsilon^{ilk} e_i \int d^3 r \, r_l E_j \partial_j A_k \\ &= \frac{1}{\mu_0 c^2} \int d^3 r \, E_k (\mathbf{r} \times \nabla) A_k + \frac{1}{\mu_0 c^2} \epsilon^{ilk} e_i \int d^3 r \, E_l A_k, \end{aligned} \tag{7.27-5}$$

which followed from a second IBP and dropping of the boundary term. This can be written in total as

$$\boxed{\mathbf{L} = \frac{1}{\mu_0 c^2} \int d^3 r \, \left[\mathbf{E} \times \mathbf{A} + E_k (\mathbf{r} \times \nabla) A_k \right]} \tag{7.27-6}$$

as desired. Note that the sum over k is implicit.

(b) Consider an expansion of the vector potential in the radiation gauge in terms of plane waves:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\lambda} \int \frac{d^3 k}{(2\pi)^3} \left[\varepsilon_\lambda (\mathbf{k}) a_\lambda (\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t} + \text{c.c} \right]. \tag{7.27-7}$$

The polarization vectors $\boldsymbol{\varepsilon}_\lambda(\mathbf{k})$ are conveniently chosen as the positive and negative helicity vectors $\boldsymbol{\varepsilon}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2)$ where $\boldsymbol{\varepsilon}_1, \ \boldsymbol{\varepsilon}_2 \in \mathbb{R}$ are orthogonal vectors in the plane whose positive normal is in the direction of $\mathbf{k}$.

Show that the time average of the first (spin) term of $\mathbf{L}$ can be written as

$$\mathbf{L}_{\text{spin}} = \frac{2}{\mu_0 c}\int \frac{d^3k}{(2\pi)^3}\mathbf{k}\left[|a_+(\mathbf{k})|^2 - |a_-(\mathbf{k})|^2\right]. \tag{7.27-8}$$

Can the term "spin" angular momentum be justified from this expression? Calculate the energy of the field in terms of the plane wave expansion of $\mathbf{A}$ and compare.

Solution: The vector potential can be written

$$\mathbf{A}(\mathbf{r},t) = \int \frac{d^3k'}{(2\pi)^3}\left[(\boldsymbol{\varepsilon}_+(\mathbf{k}')a_+(\mathbf{k}') + \boldsymbol{\varepsilon}_-(\mathbf{k}')a_-(\mathbf{k}'))\, e^{i\mathbf{k}'\cdot\mathbf{x}-i\omega' t} + \left(\boldsymbol{\varepsilon}_-(\mathbf{k}')a_+^*(\mathbf{k}') + \boldsymbol{\varepsilon}_+(\mathbf{k}')a_-^*(\mathbf{k}')\right) e^{-i\mathbf{k}'\cdot\mathbf{x}+i\omega' t}\right], \tag{7.27-9}$$

where I expanded the sum over λ and noted that $\boldsymbol{\varepsilon}_+^* = \boldsymbol{\varepsilon}_-$. The electric field follows from $\mathbf{E} = -\dot{\mathbf{A}}$:

$$\mathbf{E}(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3}(i\omega)\left[(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k}))\, e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} - \left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right) e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right], \tag{7.27-10}$$

which implies that

$$\begin{aligned}\mathbf{E}\times\mathbf{A} = \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}(i\omega)\Big[&(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k}))\times(\boldsymbol{\varepsilon}_+(\mathbf{k}')a_+(\mathbf{k}') + \boldsymbol{\varepsilon}_-(\mathbf{k}')a_-(\mathbf{k}'))\, e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}-i(\omega+\omega')t}\\
&+ (\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k}))\times\left(\boldsymbol{\varepsilon}_-(\mathbf{k}')a_+^*(\mathbf{k}') + \boldsymbol{\varepsilon}_+(\mathbf{k}')a_-^*(\mathbf{k}')\right) e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}-i(\omega-\omega')t}\\
&- \left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\times(\boldsymbol{\varepsilon}_+(\mathbf{k}')a_+(\mathbf{k}') + \boldsymbol{\varepsilon}_-(\mathbf{k}')a_-(\mathbf{k}'))\, e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}+i(\omega-\omega')t}\\
&- \left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\times\left(\boldsymbol{\varepsilon}_-(\mathbf{k}')a_+^*(\mathbf{k}') + \boldsymbol{\varepsilon}_+(\mathbf{k}')a_-^*(\mathbf{k}')\right) e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}+i(\omega+\omega')t}\Big].\end{aligned} \tag{7.27-11}$$

When this is substituted into (7.27-6), the integral over all of space swaps the exponentials to delta functions which collapse one of the momentum integrals,

$$\begin{aligned}\mathbf{L}_{\text{spin}} &= \frac{1}{\mu_0 c^2}\int d^3r\, \mathbf{E}\times\mathbf{A}\\
&= \frac{1}{\mu_0 c^2}\int \frac{d^3k}{(2\pi)^3}(i\omega)\Big[(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k}))\times(\boldsymbol{\varepsilon}_+(-\mathbf{k})a_+(-\mathbf{k}) + \boldsymbol{\varepsilon}_-(-\mathbf{k})a_-(-\mathbf{k}))\, e^{-2i\omega t}\\
&+ (\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k}))\times\left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\\
&- \left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\times(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k}))\\
&- \left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\times\left(\boldsymbol{\varepsilon}_-(-\mathbf{k})a_+^*(-\mathbf{k}) + \boldsymbol{\varepsilon}_+(-\mathbf{k})a_-^*(-\mathbf{k})\right) e^{2i\omega t}\Big].\end{aligned} \tag{7.27-12}$$

If I now time average the expression, since $\langle\exp(\pm 2i\omega t)\rangle = 0$, only two terms remain,

$$\begin{aligned}\langle\mathbf{L}\rangle_{\text{spin}} = \frac{1}{\mu_0 c^2}\int \frac{d^3k}{(2\pi)^3}(i\omega)\Big[&(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k}))\times\left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\\
&- \left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\times(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k}))\Big],\end{aligned} \tag{7.27-13}$$

where since $\boldsymbol{\varepsilon}_+\times\boldsymbol{\varepsilon}_- = -i\hat{k}$ and all else are zero,

$$\begin{aligned}\langle\mathbf{L}\rangle_{\text{spin}} &= \frac{1}{\mu_0 c^2}\int \frac{d^3k}{(2\pi)^3}(i\omega)\Big[-i\hat{k}|a_+(\mathbf{k})|^2 + i\hat{k}|a_-(\mathbf{k})|^2 - i\hat{k}|a_+(\mathbf{k})|^2 + i\hat{k}|a_-(\mathbf{k})|^2\Big]\\
&= \frac{2}{\mu_0 c^2}\int \frac{d^3k}{(2\pi)^3}(\omega\hat{k})\Big[|a_+(\mathbf{k})|^2 - |a_-(\mathbf{k})|^2\Big].\end{aligned} \tag{7.27-14}$$

Lastly, $\omega\hat{k} = ck\hat{k} = c\mathbf{k}$,

$$\boxed{\langle\mathbf{L}\rangle_{\text{spin}} = \frac{2}{\mu_0 c}\int \frac{d^3k}{(2\pi)^3}\mathbf{k}\left[|a_+(\mathbf{k})|^2 - |a_-(\mathbf{k})|^2\right]} \tag{7.27-15}$$

which is the desired expression. Is this truly a spin angular momentum? Here,

$$\mathbf{E}\cdot\mathbf{E}^* = \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\omega\omega' \left[\left(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k})\right)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} - \left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right]$$
$$\cdot\left[\left(\boldsymbol{\varepsilon}_+^*(\mathbf{k}')a_+^*(\mathbf{k}') + \boldsymbol{\varepsilon}_-^*(\mathbf{k}')a_-^*(\mathbf{k}')\right)e^{-i\mathbf{k}'\cdot\mathbf{x}+i\omega' t} - \left(\boldsymbol{\varepsilon}_-^*(\mathbf{k}')a_+^*(\mathbf{k}') + \boldsymbol{\varepsilon}_+^*(\mathbf{k}')a_-^*(\mathbf{k}')\right)e^{i\mathbf{k}'\cdot\mathbf{x}-i\omega' t}\right], \tag{7.27-16}$$

which results in

$$\begin{aligned}\frac{1}{\mu_0 c^2}\int d^3r|\mathbf{E}|^2 = \frac{1}{\mu_0 c^2}\int \frac{d^3k}{(2\pi)^3}\omega^2\Big[&\left(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k})\right)\cdot\left(\boldsymbol{\varepsilon}_+^*(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_-^*(\mathbf{k})a_-^*(\mathbf{k})\right)\\
&-\left(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) - \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k})\right)\cdot\left(\boldsymbol{\varepsilon}_-^*(-\mathbf{k})a_+^*(-\mathbf{k}) + \boldsymbol{\varepsilon}_+^*(-\mathbf{k})a_-^*(-\mathbf{k})\right)e^{-2i\omega t}\\
&-\left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\cdot\left(\boldsymbol{\varepsilon}_+^*(-\mathbf{k})a_+^*(-\mathbf{k}) + \boldsymbol{\varepsilon}_-^*(-\mathbf{k})a_-^*(-\mathbf{k})\right)e^{2i\omega t}\\
&+\left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\cdot\left(\boldsymbol{\varepsilon}_-^*(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+^*(\mathbf{k})a_-^*(\mathbf{k})\right)\Big]\\
\frac{1}{\mu_0 c^2}\int d^3r\left\langle|\mathbf{E}|^2\right\rangle = \frac{2}{\mu_0 c^2}\int \frac{d^3k}{(2\pi)^3}\omega^2\Big[&|a_+(\mathbf{k})|^2 + |a_-(\mathbf{k})|^2\Big],\end{aligned} \tag{7.27-17}$$

where I used $\boldsymbol{\varepsilon}_\pm^*\cdot\boldsymbol{\varepsilon}_\pm = 1$, else are zero. This is the energy of the electric field, and the energy of the magnetic field can be found via

$$\begin{aligned}\frac{1}{\mu_0}\int d^3r\mathbf{J}\cdot\mathbf{A} = \frac{1}{\mu_0}\int d^3r(\nabla\times\mathbf{B})\cdot\mathbf{A} &= \frac{1}{\mu_0}\int d^3r(\nabla\times(\nabla\times\mathbf{A}))\cdot\mathbf{A}\\
&= -\frac{1}{\mu_0}\int d^3r\;\nabla^2\mathbf{A}\cdot\mathbf{A},\end{aligned} \tag{7.27-18}$$

which follows from Coulomb gauge. Here,

$$\nabla^2\mathbf{A} = -\int \frac{d^3k}{(2\pi)^3}k^2\left[\left(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k})\right)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + \left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right], \tag{7.27-19}$$

which follows that

$$\nabla^2\mathbf{A}\cdot\mathbf{A} = -\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}k^2\left[\left(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k})\right)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + \left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right]$$
$$\cdot\left[\left(\boldsymbol{\varepsilon}_+(\mathbf{k}')a_+(\mathbf{k}') + \boldsymbol{\varepsilon}_-(\mathbf{k}')a_-(\mathbf{k}')\right)e^{i\mathbf{k}'\cdot\mathbf{x}-i\omega' t} + \left(\boldsymbol{\varepsilon}_-(\mathbf{k}')a_+^*(\mathbf{k}') + \boldsymbol{\varepsilon}_+(\mathbf{k}')a_-^*(\mathbf{k}')\right)e^{-i\mathbf{k}'\cdot\mathbf{x}+i\omega' t}\right] \tag{7.27-20}$$

$$\begin{aligned}\int d^3r\nabla^2\mathbf{A}\cdot\mathbf{A} = -\int \frac{d^3k}{(2\pi)^3}k^2\Big[&\left(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k})\right)\cdot\left(-\boldsymbol{\varepsilon}_+(-\mathbf{k})a_+(\mathbf{k}) - \boldsymbol{\varepsilon}_-(-\mathbf{k})a_-(\mathbf{k})\right)e^{-2i\omega t}\\
&+2\left(\boldsymbol{\varepsilon}_+(\mathbf{k})a_+(\mathbf{k}) + \boldsymbol{\varepsilon}_-(\mathbf{k})a_-(\mathbf{k})\right)\cdot\left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\\
&+\left(\boldsymbol{\varepsilon}_-(\mathbf{k})a_+^*(\mathbf{k}) + \boldsymbol{\varepsilon}_+(\mathbf{k})a_-^*(\mathbf{k})\right)\cdot\left(\boldsymbol{\varepsilon}_-(\mathbf{k}')a_+^*(\mathbf{k}') + \boldsymbol{\varepsilon}_+(\mathbf{k}')a_-^*(\mathbf{k}')\right)e^{2i\omega t}\\
-\frac{1}{\mu_0}\int d^3r\left\langle\nabla^2\mathbf{A}\cdot\mathbf{A}\right\rangle = \frac{2}{\mu_0}\int \frac{d^3k}{(2\pi)^3}k^2\Big[&|a_+(\mathbf{k})|^2 + |a_-(\mathbf{k})|^2\Big].\end{aligned} \tag{7.27-21}$$

I therefore find that the time averaged energy has the form

$$W = \frac{1}{2}\left(\frac{1}{\mu_0 c^2}\int d^3r\left\langle|\mathbf{E}|^2\right\rangle - \frac{1}{\mu_0}\int d^3r\left\langle\nabla^2\mathbf{A}\cdot\mathbf{A}\right\rangle\right) = \frac{2}{\mu_0 c}\int \frac{d^3k}{(2\pi)^3}\omega k\Big[|a_+(\mathbf{k})|^2 + |a_-(\mathbf{k})|^2\Big] \tag{7.27-22}$$

Comparing this with (7.27-15) I immediately conclude that

$$\boxed{W_+ = \omega|\mathbf{L}_{\text{spin},+}|} \qquad\qquad \boxed{W_- = \omega|\mathbf{L}_{\text{spin},-}|} \tag{7.27-23}$$

where I have decomposed $W = W_+ + W_-$ and similarly for $\mathbf{L}_{\text{spin}}$. Thus, the energy is the angular frequency times the angular momentum norm, which is precisely $E = hf = \hbar\omega$ for photons since $L = \hbar$, where $\hbar$ is the intrinsic spin of the photon. I conclude that the quantity (7.27-15) is representative of the intrinsic spin of the EM field.

Chapter 8

Jackson (8.2)

A transmission line consisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ, as shown, is filled with a uniform lossless dielectric μ, ϵ. A TEM mode is propagated along this line. Section 8.1 applies.

(a) Show that the time-averaged power flow along the line is

$$P = \sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right), \tag{8.2-1}$$

where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor.

Solution: Since the TEM-mode is also a solution to an electrostatic problem, I can solve this as a static problem then tag on exponentials that make this a wave-guide problem. Let there be a total charge per unit length λ lie on the inner conductor, then by Gauss Law I find that $\mathbf{E} = \frac{\lambda}{2\pi\epsilon s}\hat{s}$. This relates to the magnetic induction via the relation $\mathbf{B} = \sqrt{\mu\epsilon}\hat{z} \times \mathbf{E} = \sqrt{\frac{\mu}{\epsilon}}\frac{\lambda}{2\pi s}\hat{\phi}$. The peak of the magnetic field at the inner surface $s = a$ then follows as

$$H_0 = \frac{1}{\sqrt{\mu\epsilon}}\frac{\lambda}{2\pi a} \iff \lambda = 2\pi a\sqrt{\mu\epsilon}H_0, \tag{8.2-2}$$

and the static problem has the solution

$$\mathbf{H}_{\text{static}} = \frac{a}{s}H_0\hat{\phi}, \qquad \mathbf{E}_{\text{static}} = \sqrt{\frac{\mu}{\epsilon}}\frac{a}{s}H_0\hat{s}. \tag{8.2-3}$$

The general TEM mode solution then has the form

$$\mathbf{H}_{\text{TEM}} = \frac{a}{s}H_0 e^{i(kz-\omega t)}\hat{\phi}, \qquad \mathbf{E}_{\text{TEM}} = \sqrt{\frac{\mu}{\epsilon}}\frac{a}{s}H_0 e^{i(kz-\omega t)}\hat{s}, \tag{8.2-4}$$

The time averaged Poynting vector then follows from $\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$, so

$$\mathbf{S} = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\frac{a^2}{s^2}|H_0|^2 \hat{s} \times \hat{\phi} = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\frac{a^2}{s^2}|H_0|^2 \hat{z}, \tag{8.2-5}$$

which can be integrated to give the time-averaged power flow:

$$P = \int d\mathbf{a} \cdot \mathbf{S} = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}a^2|H_0|^2 \int_0^{2\pi} d\phi \int_a^b \frac{ds}{s} \implies \boxed{P = \sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)} \tag{8.2-6}$$

as desired.

(b) Show that the transmitted power is attenuated along the line as

$$P(z) = P_0 e^{-2\gamma z}, \qquad \gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln(b/a)}. \tag{8.2-7}$$

Solution: Section 8.1 applies here, so the power loss is of the form

$$\frac{dP_{\text{loss}}}{da} = \frac{\mu_c \omega\delta}{4}|\mathbf{H}|^2, \tag{8.2-8}$$

where μ_c is the conductor permeability and $\delta = \sqrt{2/\mu_c\omega\sigma}$ is its skin depth. The mode is dying out along the direction of propagation z, so project out this power loss from (8.2-8):

$$\begin{aligned}\frac{dP_{\text{loss}}}{dz} &= \frac{\mu_c\omega\delta}{4}\int_0^{2\pi} d\phi\ \left(a|\mathbf{H}(s=a)|^2 + b|\mathbf{H}(s=b)|^2\right) \\ &= \frac{\pi\mu_c\omega\delta}{2}\left(a|\mathbf{H}(s=a)|^2 + b|\mathbf{H}(s=b)|^2\right),\end{aligned} \tag{8.2-9}$$

where I noted that the loss is attributed to interactions with the inner and outer surfaces. Using (8.2-4),

$$\begin{aligned}\frac{dP_{\text{loss}}}{dz} &= \frac{\pi\mu_c\omega\delta}{2}\left(a|H_0|^2 + \frac{a^2}{b}|H_0|^2\right) \\ &= \frac{\pi a^2}{\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)|H_0|^2.\end{aligned} \tag{8.2-10}$$

Now, this is the power loss, so there is the relation $dP/dz = -dP_{\text{loss}}/dz$. Assume that the change in power is proportional to the power itself: $dP/dz = -2\gamma P$, which is to align with the problem statement,

$$\boxed{P(z) = P_0 e^{-2\gamma z}} \tag{8.2-11}$$

all that remains is to determine γ. By construction,

$$\gamma = \frac{1}{2P}\frac{\partial P_{\text{loss}}}{\partial z} = \frac{\frac{\pi a^2}{\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)|H_0|^2}{2\sqrt{\frac{\mu}{\epsilon}}\pi a^2|H_0|^2\ln\left(\frac{b}{a}\right)} \implies \boxed{\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln(b/a)}} \tag{8.2-12}$$

as desired.

(c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position z. Show that for this line,

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\left(\frac{b}{a}\right) \tag{8.2-13}$$

Solution: The voltage difference between the cylinders is easily obtained from the electric field (8.2-4),

$$\Delta\Phi = \int_a^b d\mathbf{s}\cdot\mathbf{E} = \sqrt{\frac{\mu}{\epsilon}}aH_0 e^{i(kz-\omega t)}\int_a^b \frac{ds}{s} = \sqrt{\frac{\mu}{\epsilon}}aH_0e^{i(kz-\omega t)}\ln\left(\frac{b}{a}\right), \tag{8.2-14}$$

and the axial (surface) current is easily determined through

$$\mathbf{K} = \hat{s}\times\mathbf{H}\Big|_{s=a} = H_0 e^{i(kz-\omega t)}\hat{z}. \tag{8.2-15}$$

Since I want the current I, my choice in inner/outer cylinder is irrelevant. This current is simply

$$I = \int dl_\perp\ K = \int_0^{2\pi} a d\phi\ \left(H_0 e^{i(kz-\omega t)}\right) = 2\pi a H_0 e^{i(kz-\omega t)}, \tag{8.2-16}$$

where since the characteristic impedance is defined as the ratio of (8.2-14) to (1.9), I find

$$\boxed{Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \ln\left(\frac{b}{a}\right)} \tag{8.2-17}$$

as desired.

(d) Show that the series resistance and inductance per unit length of the line are

$$\mathcal{R} = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right) \qquad \mathcal{L} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right), \tag{8.2-18}$$

where μ_c is the permeability of the conductor. The correction to the inductance comes from the penetration of the flux into the conductors by a distance of order δ.

Solution: The energy loss dP_{loss}/dz is manifest as Joule heating: $P_{\text{loss}} = I_{\text{rms}}^2 R = \frac{1}{2}I^2 R$. The resistance per unit length is $\mathcal{R} = dR/dz$, so

$$\mathcal{R} = \frac{2}{I^2}\frac{dP_{\text{loss}}}{dz} = \frac{2\pi a^2}{I^2\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)|H_0|^2, \tag{8.2-19}$$

where according to (8.2-16), $H_0 = I/2\pi a$ and I find

$$\boxed{\mathcal{R} = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)} \tag{8.2-20}$$

as the desired resistance per unit length. The inductance per unit length can be determined through the relation $W = \frac{1}{2}LI_{\text{rms}}^2 = \frac{1}{4}LI^2$, from which

$$\mathcal{L} = \frac{4}{I^2}\frac{\partial W}{\partial z} = \frac{4}{I^2}\left[\frac{\partial W_{s<a}}{\partial z} + \frac{\partial W_{a<s<b}}{\partial z} + \frac{\partial W_{s>b}}{\partial z}\right], \tag{8.2-21}$$

where I am accounting for the fact that the magnetic fields decay inside the conductors according to the skin depth: $\mathbf{H}_c = \mathbf{H}_{\|}e^{-\xi/\delta}$, where ξ is the distance relative to the boundary of the conductor. Using the magnetic field (8.2-4),

$$\begin{aligned}
\mathcal{L} &= \frac{4}{I^2}\left[\frac{1}{4\mu_c}(2\pi)\int_0^a sds|\mathbf{H}_{s<a}|^2 + \frac{1}{4\mu}(2\pi)\int_a^b sds|\mathbf{H}_{a<s<b}|^2 + \frac{1}{4\mu_c}(2\pi)\int_b^\infty sds|\mathbf{H}_{s>b}|^2\right] \\
&= \frac{2\pi}{I^2}|H_0|^2\left[\mu_c\int_0^a se^{-2(a-s)/\delta}ds + \mu a^2\int_a^b \frac{ds}{s} + \frac{\mu_c a^2}{b^2}\int_b^\infty se^{-2(s-b)/\delta}ds\right] \\
&= \frac{1}{2\pi a^2}\left[\mu_c e^{-2a/\delta}\int_0^a se^{2s/\delta}ds + \mu a^2\ln\left(\frac{b}{a}\right) + \frac{\mu_c a^2}{b^2}e^{2b/\delta}\int_b^\infty se^{-2s/\delta}ds\right] \\
&= \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c}{8\pi}\left[2\delta\left(\frac{1}{a}+\frac{1}{b}\right) - \left(\frac{\delta^2}{a^2}-\frac{\delta^2}{b^2}\right) + \frac{\delta^2}{a^2}e^{-2a/\delta}\right],
\end{aligned} \tag{8.2-22}$$

Since $\delta \ll a < b$, to order δ I find the desired self-inductance per unit length,

$$\boxed{\mathcal{L} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)} \tag{8.2-23}$$

Jackson (8.5) "incomplete"

A waveguide is constructed so that the cross section of the guide forms a right triangle with sides of length $a, a, \sqrt{2}a$, as shown. The medium inside has $\mu_r = \epsilon_r = 1$.

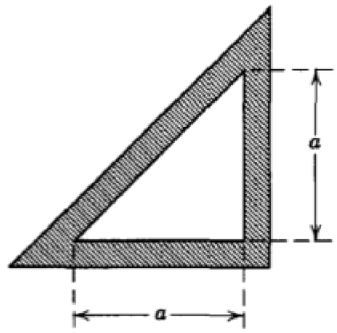


Figure 10: Jackson (8.5)

(a) Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.

Solution: The possible modes of propagation can only be TE or TM, as TEM modes are forbidden with this being a single conductor. I will determine both of these modes from relation Eq.(8.41) from Jackson,

$$\left(\partial_x^2 + \partial_y^2 + \gamma_{nm}^2\right)\psi_{nm} = 0, \tag{8.5-1}$$

where $\psi = H_z$ (TE) and $\psi = E_z$ (TM). The boundary conditions are $\left.\frac{\partial\psi}{\partial n}\right|_{\text{surf.}} = 0$ (TE) or $\psi|_{\text{surf.}} = 0$ on the three surfaces $y = 0,\ x = a,\ y = x$. Assume permeability and permittivity μ, ϵ. The trick in solving this problem to take inspiration from section 8.4 of Jackson in which he solves the problem for a square,

$$\text{(square)} \qquad \psi_{nm}^{\text{TE}} = H_0 \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right), \qquad \psi_{nm}^{\text{TM}} = E_0 \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) \tag{8.5-2}$$

which indeed satisfy their appropriate boundary conditions on the four surfaces of a square of side length a. I can generate, through particular linear combinations, solutions which are either even or odd with respect to the $y = x$ boundary surface. This has the effect of generating a (TE) or (TM) boundary condition on the $y = x$ line for the square modes, which describes precisely the problem at hand.

For the Dirichlet BC (TM), the linear combination must be odd under the transformation $y \leftrightarrow x$ to satisfy $\psi|_{y=x} = 0$, and for the Neumann BC (TE), the linear combination must be even to satisfy $\left.\frac{\partial\psi}{\partial n}\right|_{y=x} = 0$ (derivative picks up the relative minus). This naturally leads to the choices

$$\begin{aligned}
\psi_{nm}^{\text{TE}} &= H_0\left[\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right) + \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)\right] \\
\psi_{nm}^{\text{TM}} &= E_0\left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) - \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\right]
\end{aligned} \tag{8.5-3}$$

both of which satisfy the BC's of a square waveguide AND the right-triangular waveguide above:

$$\begin{aligned}
\left.\psi_{nm}^{\text{TM}}\right|_{y=x} &= E_0\left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) - \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\right] = 0 \\
\left.\hat{n}\cdot\nabla\psi_{nm}^{\text{TE}}\right|_{y=x} &= \left.\partial_x\psi_{nm}^{\text{TM}}\right|_{y=x} - \left.\partial_y\psi_{nm}^{\text{TM}}\right|_{y=x} \\
&= H_0\Big[-\frac{n\pi}{a}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right) - \frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) \\
&\qquad + \frac{m\pi}{a}\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) + \frac{n\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\Big]_{y=x} \\
&= 0.
\end{aligned} \tag{8.5-4}$$

Hence, there are two possible (TE) or (TM) modes for the right-triangle:

$$\boxed{H_z = H_0 \left[\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right) + \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)\right]} \tag{8.5-5}$$

$$\boxed{E_z = E_0 \left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) - \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\right]} \tag{8.5-6}$$

Both of these can be substituted into (8.5-1) to determine the γ's:

$$\begin{aligned}
-\gamma_{nm}^2\psi_{nm}^{\text{TE}} &= (\partial_x^2 + \partial_y^2)\psi_{nm}^{\text{TE}} \\
&= H_0\Big[-\left(\frac{n\pi}{a}\right)^2 \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right) - \left(\frac{m\pi}{a}\right)^2 \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right) \\
&\quad - \left(\frac{m\pi}{a}\right)^2 \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right) - \left(\frac{n\pi}{a}\right)^2 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)\Big] \\
&= -\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]\psi_{nm}^{\text{TE}} \implies \gamma_{nm}^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2, \\
-\gamma_{nm}^2\psi_{nm}^{\text{TM}} &= (\partial_x^2 + \partial_y^2)\psi_{nm}^{\text{TM}} \\
&= H_0\Big[-\left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) + \left(\frac{m\pi}{a}\right)^2 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \\
&\quad \left(\frac{m\pi}{a}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) - \left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\Big] \\
&= -\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]\psi_{nm}^{\text{TM}} \implies \gamma_{nm}^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2.
\end{aligned} \tag{8.5-7}$$

which are the same for (TE) or (TM). Since (8.38) of Jackson says $\omega_{nm} = \gamma_{nm}/\sqrt{\mu\epsilon}$, I find the cutoff frequencies

$$\boxed{\omega_{nm} = \frac{1}{\sqrt{\mu\epsilon}}\frac{\pi}{a}\sqrt{n^2 + m^2}} \tag{8.5-8}$$

for the right-triangle.

(b) For the lowest modes of each type calculate the attenuation constant, assuming that the walls have large, but finite, conductivity. Compare the result with that for a square guide of side a made from the same material.

Solution: still needs to be finished!