Entanglement Spectrum bCFT Correspondence

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June 2025

Definitions

Matrix Product State

This is just a rewriting of a many body wavefunction

$$|\psi\rangle = \sum_{\{s\}} c_{s_1,s_2,\cdots,s_N} |s_1, s_2, \cdots, s_N\rangle = \sum_{\{s\}} A^{(s_1)} A^{(s_2)} \cdots A^{(s_N)} |s_1, s_2, \cdots, s_N\rangle\,, \tag{1}$$

where I've assumed translation invariance, all A's are the same. Here, each A is a $D \times D$ matrix, where larger D corresponds to more entanglement (coupling between spin states). For example, when $D = 1$ there is no entanglement, as the state can be written as a pure state $|\psi\rangle = a_1 |s_1\rangle \otimes a_2 |s_2\rangle \otimes \cdots \otimes a_N |s_N\rangle$, where $\{a_i\}$ are some coefficients. These matrix product state representations are useful for low entanglement systems.

Entanglement Spectrum

This notion was first devised by Li and Haldane [2]. Take a quantum system in a pure state ψ, which could be conveniently taken as the ground state of some Hamiltonian, and divide the subsystem into two parts, say left (L) or right (R). The Schmidt decomposition of such state takes the form

$$|\psi\rangle = \sum_i \lambda_i \left|\psi_i^L\right\rangle \otimes \left|\psi_i^R\right\rangle, \tag{2}$$

where $\left|\psi^i_{L/R}\right\rangle \in \mathcal{H}_{L/R}$. The density matrix for this pure state is

$$\rho = |\psi\rangle\langle\psi| = \sum_{i,j} \lambda_i \lambda_j \left|\psi_i^L\right\rangle \left\langle\psi_j^L\right| \otimes \left|\psi_i^R\right\rangle \left\langle\psi_j^R\right|, \tag{3}$$

from which the reduced density matrix $\rho_{L/R}$ can be found by tracing out the other subspace,

$$\rho_L = \mathrm{Tr}_R \rho = \sum_{i,j} \lambda_i \lambda_j \left|\psi_i^L\right\rangle \left\langle\psi_j^L\right| \otimes \mathrm{Tr}[\left|\psi_i^R\right\rangle \left\langle\psi_j^R\right|] = \sum_{i,j} \lambda_i \lambda_j \left|\psi_i^L\right\rangle \left\langle\psi_j^L\right| \otimes \delta_{ij}\mathbb{1} = \sum_i \lambda_i^2 \left|\psi_i^L\right\rangle \left\langle\psi_i^L\right| \otimes \mathbb{1}, \tag{4}$$

similarly for ρ_R. This reduced density matrix encodes the entanglement between the L/R subsystems; along this line it is important to note that while $|\psi\rangle$ is pure on $\mathcal{H}_L \otimes \mathcal{H}_R$, after tracing out a subspace, the reduced state in the remaining state can be mixed. It doesn't matter whether we determine the entanglement spectrum (to be defined) from ρ_L or ρ_R since they have the same spectrum. We define the entanglement Hamiltonian through

$$\rho_L = \exp(-H_E) \iff H_E \propto -\ln \rho_L. \tag{5}$$

The spectrum of the entanglement Hamiltonian, ε_i, is known as the entanglement spectrum:

$$\exp(-H_E) = \sum_i e^{-\varepsilon_i} |\psi_i\rangle\langle\psi_i| \iff e^{-\varepsilon_i} = \lambda_i^2 \iff \varepsilon_i = -\ln \lambda_i^2. \tag{6}$$

Sometimes this is defined differently, but this way of writing it flows naturally from the Schmidt decomposition of a state The actual entanglement (von-Neumann) entropy is

$$S = -\mathrm{Tr}[\rho_L \ln \rho_L] = -\sum_i \lambda_i \ln \lambda_i = \frac{1}{2} \sum_i \varepsilon_i e^{-\varepsilon_i/2}. \tag{7}$$

In many sources [4, 3], the entanglement spectrum is plotted as an array, where the horizontal axis is some U(1) charge and the vertical axis is the magnitude. Whenever there is a symmetry imposed, blocks of ρ_L can be labeled by some fixed U(1), and we can associate subsets of the full entanglement spectrum $\{\varepsilon_i\}$ with those of different fixed U(1) charge: $\{\varepsilon_i\} = \{\varepsilon_i\}_{q=0} \cup \{\varepsilon_i\}_{q=1} \cup \cdots$.

Getting Luttinger Liquid Parameters from ES

It was observed in [3] that the low-lying part of the E.S. for a bipartition of the Matrix-Product-State (MPS) resembles the energy spectrum of a boundary conformal field theory (CFT). The fact that this energy spectrum depends explicitly on the scaling dimension Δ of the theory, a consequence of the state-operator correspondence, allows for a direct determination of the phase boundary $\Delta = 2$. This is typically done in the Luttinger-liquid formalism, where the Luttinger-liquid parameter K is directly related to the scaling dimension. For our purposes, we can think of the Luttinger-liquid effective (1D-quantum) Hamiltonian as describing two fields, one of which being that of the OG XY model, and the other being that of the dual sine-Gordon model. This is very much in line with the analytic approach used in [1], determining the scaling dimensions by mapping to a sine-Gordon theory.

Getting Luttinger Liquid Parameters from ES

Forewarning that there are many different convention for the Luttinger-liquid model, from CM to high-energy theory. The paper [3] investigates 1D quantum systems on chains of length L with OBC or PBC, getting the E.S. from the ground state wavefunction determined through DMRG. It is known in a 1+1D CFT on a strip of length L that the energy spectrum is directly proportional to the scaling dimension Δ,

$$E = \frac{2\pi v}{L}\left(\Delta - \frac{c}{24}\right), \tag{8}$$

where v is the velocity of excitations (typically $v = 1$), and c is the central charge of the theory ($c = 1$ for bosons, $c = 1/2$ for fermions) whose contribution is nothing but the Casimir energy on the strip.

I will quote Δ for the Luttinger-liquid Hamiltonian quoted in the tensor-network paper [4],

$$H = \frac{v}{2\pi}\int dx \left(K(\nabla\theta)^2 + \frac{1}{K}(\nabla\phi)^2\right) + g\int dx\ \cos(2\phi). \tag{9}$$

Clearly, θ can be associated with the spin-wave fluctuations of a XY model, and the ϕ is the dual field in the sine-Gordon theory. To see this, in the vortex sector of (9) we have for $v = 1$ and $K \equiv \pi\beta J$,

$$\int dx\ \left[\frac{1}{2\pi}\frac{1}{\pi\beta J}(\nabla\phi)^2 + g\cos(2\phi)\right] \to \int dx\ \left[\frac{1}{2\beta J}(\nabla\phi)^2 + g\cos(2\pi\phi)\right], \tag{10}$$

where from our previous analysis, the scaling dimension of the vertex operator is $\Delta = \pi\beta J = K$, with $K = 2$ being the transition. For the convention (9), it is true for a general vertex operator $\mathcal{O} = \exp(in\theta + im\phi)$ that

$$\Delta = \Delta_{n,m} + N = \frac{n^2}{4K} + \frac{Km^2}{4} + N, \tag{11}$$

where N is the oscillator level, n is the U(1) charge, and m is related to vortex winding. Hence,

$$E_{n,m,N} = E_{0,0,0} + \frac{2\pi v}{L}\left(\frac{n^2}{4K} + \frac{Km^2}{4} + N\right). \tag{12}$$

Notice how the E.S. in [4, 3] is quadratic in the U(1) charge, just like the energy spectrum (12). Since these numerical studies work with the U(1) charge sector, if we want the Luttinger parameter, we can compute it from the ratio

$$\frac{E_{1,0,0} - E_{0,0,0}}{E_{0,0,1} - E_{0,0,0}} = \frac{1}{4K} \xrightarrow{\text{XY}} \frac{1}{4\pi\beta J} = \frac{\eta}{2}, \tag{13}$$

where η is the critical exponent for the spin-spin correlations in the original XY model, defined to be twice the associated scaling dimension. With η known, the phase transition occurs for $\eta = 1/4$ for the KT universality class. Making the correspondence to the E.S., this ratio method can be applied directly to the E.S. as well, with an example shown in Fig.[1].

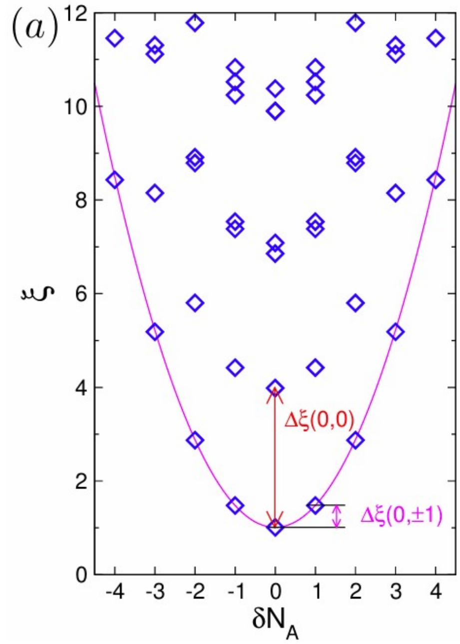

Figure 1: Sample E.S. taken from [3]. The entanglement spectrum lives on the vertical axis for each U(1) charge on the horizontal axis. The quantity $\Delta\xi(0,\pm 1)$ is the numerator of (13), while $\Delta\xi(0,0)$ is the denominator.

Our Model

I will now write an XY model which limits to our nLSM

$$S = \beta \int_x \Big[\frac{\kappa_1}{2} \left((\partial_x\theta)^2 + (\partial_y\overline{\theta})^2 \right) + \frac{\kappa_2}{2} \left((\partial_y\theta)^2 + (\partial_x\overline{\theta})^2 \right) + \frac{\kappa_1'}{2} \left((\partial_x\phi)^2 + (\partial_y\overline{\phi})^2 \right) + \frac{\kappa_2'}{2} \left((\partial_y\phi)^2 + (\partial_x\overline{\phi})^2 \right) \\ + \kappa_3 \partial_i\theta\partial_i\overline{\theta} + \kappa_4\partial_x\phi\partial_y\overline{\phi} + \kappa_5\partial_y\phi\partial_x\overline{\phi} + V\cos(2\theta - 2\overline{\theta}) \Big], \tag{14}$$

in the continuum. I have taken the stiffnesses in the θ and ϕ sectors to be distinct, and incorporated all terms allowed by symmetry.

0.1 Method-1 (HVSD Absent)

Our PDW phases are non-compact variables, and so the trig functions should look like $(\partial_x\theta)^2 \to \cos(2\Delta_x\theta)$. The gradient sector without mixed terms is simple to write,

$$\exp(-S_{\text{no-mix}}) = \prod_{\langle ij\rangle_x} \exp\Big[\frac{\beta\kappa_1}{4}\cos(2(\theta_i-\theta_j)) + \frac{\beta\kappa_2}{4}\cos(2(\overline{\theta}_i-\overline{\theta}_j)) + \frac{\beta\kappa_1'}{4}\cos(2(\phi_i-\phi_j)) + \frac{\beta\kappa_2'}{4}\cos(2(\overline{\phi}_i-\overline{\phi}_j))\Big] \\ \times \prod_{\langle ij\rangle_y} \exp\Big[\frac{\beta\kappa_2}{4}\cos(2(\theta_i-\theta_j)) + \frac{\beta\kappa_1}{4}\cos(2(\overline{\theta}_i-\overline{\theta}_j)) + \frac{\beta\kappa_2'}{4}\cos(2(\phi_i-\phi_j)) + \frac{\beta\kappa_1'}{4}\cos(2(\overline{\phi}_i-\overline{\phi}_j))\Big], \tag{15}$$

where $\langle ij\rangle_\mu$ indicates nearest neighbors along direction $\mu = x, y$, and each product acts only on the exponential it is to the left of. Before getting to the more complicated mixed-terms, the V-term is pretty simple to incorporate,

$$\exp(-S_{\text{no-mix,V}}) = \prod_{\langle ij\rangle_x} \exp\Big[\frac{\beta\kappa_1}{4}\cos(2(\theta_i-\theta_j)) + \frac{\beta\kappa_2}{4}\cos(2(\overline{\theta}_i-\overline{\theta}_j)) + \frac{\beta\kappa_1'}{4}\cos(2(\phi_i-\phi_j)) + \frac{\beta\kappa_2'}{4}\cos(2(\overline{\phi}_i-\overline{\phi}_j))\Big] \\ \times \prod_{\langle ij\rangle_y} \exp\Big[\frac{\beta\kappa_2}{4}\cos(2(\theta_i-\theta_j)) + \frac{\beta\kappa_1}{4}\cos(2(\overline{\theta}_i-\overline{\theta}_j)) + \frac{\beta\kappa_2'}{4}\cos(2(\phi_i-\phi_j)) + \frac{\beta\kappa_1'}{4}\cos(2(\overline{\phi}_i-\overline{\phi}_j))\Big] \\ \times \prod_i \exp\Big[\beta V\cos(2(\theta_i-\overline{\theta}_i)+\pi)\Big], \tag{16}$$

and lives on every site. I absorbed the overall minus sign of the V-term as a phase shift of π, for future convenience.

The mixed terms now follow. κ_3 isn't too bad as it is isotropic, but $\kappa_{4,5}$ are a bit more involved,

$$\kappa_3: \qquad \prod_{\langle ij\rangle} \exp\Big[-\frac{\beta\kappa_3}{4}\sin(2(\theta_i-\theta_j))\sin(2(\overline{\theta}_i-\overline{\theta}_j))\Big] \tag{17}$$

$$\kappa_4: \qquad \prod_{\langle ij\rangle_x}\prod_{\langle kl\rangle_y} \exp\Big[-\frac{\beta\kappa_4}{4}\sin(2(\phi_i-\phi_j))\sin(2(\overline{\phi}_k-\overline{\phi}_l))\Big] \tag{18}$$

$$\kappa_5: \qquad \prod_{\langle ij\rangle_x}\prod_{\langle kl\rangle_y} \exp\Big[-\frac{\beta\kappa_5}{4}\sin(2(\phi_k-\phi_l))\sin(2(\overline{\phi}_i-\overline{\phi}_j))\Big] \tag{19}$$

The notation can be improved by adopting a $\prod_{\langle ij\rangle}$ for each term and incorporating directionality by a finite difference operator Δ_μ, but I am trying my best to align with the notation of [4] for the time-being. For completeness, we write the partition function into uncoupled sectors

$$\mathcal{Z} = \mathcal{Z}_{\theta,\overline{\theta}}\mathcal{Z}_{\phi,\overline{\phi}}, \tag{20}$$

where

$$\begin{aligned}\mathcal{Z}_{\theta,\overline{\theta}} = \prod_i \left(\int \frac{d\theta_i}{2\pi} \frac{d\overline{\theta}_i}{2\pi} \right) \prod_{\langle ij \rangle_x} &\exp\left[\frac{\beta\kappa_1}{4} \cos(2(\theta_i - \theta_j)) + \frac{\beta\kappa_2}{4} \cos(2(\overline{\theta}_i - \overline{\theta}_j)) \right] \\ \times \prod_{\langle ij \rangle_y} &\exp\left[\frac{\beta\kappa_2}{4} \cos(2(\theta_i - \theta_j)) + \frac{\beta\kappa_1}{4} \cos(2(\overline{\theta}_i - \overline{\theta}_j)) \right] \\ \times \prod_{\langle ij \rangle} &\exp\left[-\frac{\beta\kappa_3}{4} \sin(2(\theta_i - \theta_j)) \sin(2(\overline{\theta}_i - \overline{\theta}_j)) \right] \\ \times \prod_i &\exp\left[\beta V \cos(2(\theta_i - \overline{\theta}_i) + \pi) \right],\end{aligned} \tag{21}$$

and

$$\begin{aligned}\mathcal{Z}_{\phi,\overline{\phi}} = \prod_i \left(\int \frac{d\phi_i}{2\pi} \frac{d\overline{\phi}_i}{2\pi} \right) \prod_{\langle ij \rangle_x} &\exp\left[\frac{\beta\kappa_1'}{4} \cos(2(\phi_i - \phi_j)) + \frac{\beta\kappa_2'}{4} \cos(2(\overline{\phi}_i - \overline{\phi}_j)) \right] \\ \times \prod_{\langle ij \rangle_y} &\exp\left[\frac{\beta\kappa_2'}{4} \cos(2(\phi_i - \phi_j)) + \frac{\beta\kappa_1'}{4} \cos(2(\overline{\phi}_i - \overline{\phi}_j)) \right] \\ \times \prod_{\langle ij \rangle_x} \prod_{\langle kl \rangle_y} &\exp\left[-\frac{\beta\kappa_4}{4} \sin(2(\phi_i - \phi_j)) \sin(2(\overline{\phi}_k - \overline{\phi}_l)) \right] \\ \times \prod_{\langle ij \rangle_x} \prod_{\langle kl \rangle_y} &\exp\left[-\frac{\beta\kappa_5}{4} \sin(2(\phi_k - \phi_l)) \sin(2(\overline{\phi}_i - \overline{\phi}_j)) \right].\end{aligned} \tag{22}$$

In principle, **this un-coupling will not capture the essential physics**, as we need a way of implementing the fact that π windings in θ are bound to π windings in ϕ. I will look at this in Method-2.

The idea now would be to impose the identity

$$\exp(x \cos(\theta_i - \theta_j)) = \sum_{n=-\infty}^{\infty} I_n(x) \exp(in(\theta_i - \theta_j)). \tag{23}$$

For the mixed gradient terms this is not directly applicable. It is possible to write

$$\begin{aligned}\exp(-x \sin(a) \sin(b)) &= \exp\left(\frac{x}{2} \cos(a+b) \right) \exp\left(-\frac{x}{2} \cos(a-b) \right) \\ &= \exp\left(\frac{x}{2} \cos(a+b) \right) \exp\left(\frac{x}{2} \cos(a-b+\pi) \right),\end{aligned} \tag{24}$$

which would have the consequence of mixing $\phi, \overline{\phi}$ under the cosine. In the Fourier series representation,

$$\begin{aligned}\exp(-x \sin(a) \sin(b)) &= \sum_{n,m=-\infty}^{\infty} I_n\left(\frac{x}{2}\right) I_m\left(\frac{x}{2}\right) \exp(in(a+b) + im(a-b+\pi)) \\ &= \sum_{n,m=-\infty}^{\infty} I_n\left(\frac{x}{2}\right) I_m\left(\frac{x}{2}\right) \exp(i(n+m)a + i(n-m)b + im\pi).\end{aligned} \tag{25}$$

0.2 Method-2 (HVSD Included)

This is possibly a simpler way to write the nLSM, one which also can capture the effects of HVSDs (couple $\mathcal{Z}_{\theta,\overline{\theta}}$ with $\mathcal{Z}_{\phi,\overline{\phi}}$). This would be through writing an XY model in terms of cosines of phases of the order parameters themselves, say $\cos(\theta \pm \phi),\ \cos(\overline{\theta} \pm \overline{\phi})$. Importantly,

this approach requires that $\kappa_1 = \kappa_1'$ and $\kappa_2 = \kappa_2'$.

This would have to be a choice on our part. Assuming that $\kappa_1 = \kappa_1'$ and $\kappa_2 = \kappa_2'$, we can construct

$$
\begin{aligned}
-\frac{\kappa_1}{2}\left(\cos(\Delta_x(\theta+\phi)) + \cos(\Delta_x(\theta-\phi))\right) &\sim \frac{\kappa_1}{4}\left((\partial_x\theta + \partial_x\phi)^2 + (\partial_x\theta - \partial_x\phi)^2\right) = \frac{\kappa_1}{2}\left((\partial_x\theta)^2 + (\partial_x\phi)^2\right) \\
-\frac{\kappa_1}{2}\left(\cos(\Delta_y(\overline{\theta}+\overline{\phi})) + \cos(\Delta_y(\overline{\theta}-\overline{\phi}))\right) &\sim \frac{\kappa_1}{4}\left((\partial_y\overline{\theta} + \partial_y\overline{\phi})^2 + (\partial_y\overline{\theta} - \partial_y\overline{\phi})^2\right) = \frac{\kappa_1}{2}\left((\partial_y\overline{\theta})^2 + (\partial_y\overline{\phi})^2\right) \\
-\frac{\kappa_2}{2}\left(\cos(\Delta_y(\theta+\phi)) + \cos(\Delta_y(\theta-\phi))\right) &\sim \frac{\kappa_2}{4}\left((\partial_y\theta + \partial_y\phi)^2 + (\partial_y\theta - \partial_y\phi)^2\right) = \frac{\kappa_2}{2}\left((\partial_y\theta)^2 + (\partial_y\phi)^2\right) \\
-\frac{\kappa_2}{2}\left(\cos(\Delta_x(\overline{\theta}+\overline{\phi})) + \cos(\Delta_x(\overline{\theta}-\overline{\phi}))\right) &\sim \frac{\kappa_2}{4}\left((\partial_x\overline{\theta} + \partial_x\overline{\phi})^2 + (\partial_x\overline{\theta} - \partial_x\overline{\phi})^2\right) = \frac{\kappa_2}{2}\left((\partial_x\overline{\theta})^2 + (\partial_x\overline{\phi})^2\right)
\end{aligned}
\tag{26}
$$

which naturally incorporates the fact that π windings in $\theta, \overline{\theta}$ necessarily require π windings in $\phi, \overline{\phi}$. In this language, the non-mixed gradient sector can be written as

$$
\begin{aligned}
\exp(-S_{\text{no-mix}}) = \prod_{\langle ij\rangle_x} \exp\Bigg[&\frac{\beta\kappa_1}{2}\Big(\cos((\theta_i+\phi_i)-(\theta_j+\phi_j)) + \cos((\theta_i-\phi_i)-(\theta_j-\phi_j))\Big) \\
&+ \frac{\beta\kappa_2}{2}\Big(\cos((\overline{\theta}_i+\overline{\phi}_i)-(\overline{\theta}_j+\overline{\phi}_j)) + \cos((\overline{\theta}_i-\overline{\phi}_i)-(\overline{\theta}_j-\overline{\phi}_j))\Big)\Bigg] \\
\times \prod_{\langle ij\rangle_y} \exp\Bigg[&\frac{\beta\kappa_2}{2}\Big(\cos((\theta_i+\phi_i)-(\theta_j+\phi_j)) + \cos((\theta_i-\phi_i)-(\theta_j-\phi_j))\Big) \\
&+ \frac{\beta\kappa_1}{2}\Big(\cos((\overline{\theta}_i+\overline{\phi}_i)-(\overline{\theta}_j+\overline{\phi}_j)) + \cos((\overline{\theta}_i-\overline{\phi}_i)-(\overline{\theta}_j-\overline{\phi}_j))\Big)\Bigg].
\end{aligned}
\tag{27}
$$

While it is easy to do this for these terms, for the V-term it is not quite so, as it depends on only one sector. The only way we can write $\cos(2\theta - 2\overline{\theta})$ in terms of trig functions of combinations $\theta \pm \phi,\ \overline{\theta} \pm \overline{\phi}$ is by

$$
\cos(2\theta - 2\overline{\theta}) = \frac{(1-A^2)(1-\overline{A}^2) + 4A\overline{A}}{(1+A^2)(1+\overline{A}^2)}, \qquad A = \frac{\sin(\theta+\phi) + \sin(\theta-\phi)}{\cos(\theta+\phi) + \cos(\theta-\phi)}, \qquad \overline{A} = A(\overline{\theta}, \overline{\phi}), \tag{28}
$$

which is nonsensical.

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