The Struggling Sailor

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A sailor wishes to cross (straight across) a river of width d in a boat propelled at a constant speed u. Since the water in the canal is flowing with some speed, say $v = nu$ ($n \in \mathbb{R}$), the sailor will need to continuously turn his boat in order to end up straight across from where he started from, as the current will try to drag him away. Assuming he makes it straight across, what is the equation describing the path his boat took across the river? What would the path look like if his boat wasn't fast enough?

I have drawn this situation below for an arbitrary position $r(\theta)$ of the boat on the river,

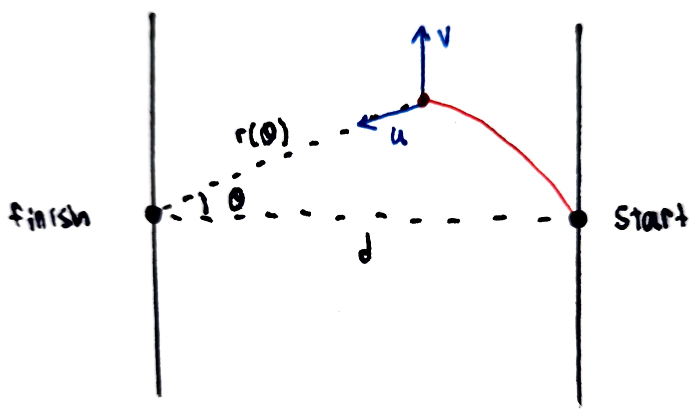

Figure 1: Velocity diagram of the boat for an arbitrary position $r(\theta)$ in the canal. The red line is the path taken, u is the velocity of the boat relative to the water, $v = un$ is the velocity of the water.

The radial coordinate of the boat is decreasing, so $\dot{r} < 0$, and the angular coordinate is increasing from $0 \to \pi/2$. My idea is to write the equations representing the radial and tangential velocities, and to use them to solve for $r(\theta)$. In the radial direction we have the constant speed u opposed by the sine component of the current:

$$-\frac{dr}{dt} = u - nu\sin\theta \iff dt = \frac{dr}{nu\sin\theta - u}, \tag{1}$$

while in the tangential direction we just have the cosine component of the current:

$$r\frac{d\theta}{dt} = nu\cos\theta \iff dt = \frac{rd\theta}{nu\cos\theta}, \tag{2}$$

where $\dot{s} = r\dot{\theta}$ is the tangential velocity at any given r. I can eliminate t by equating the two equations, giving

$$\begin{aligned} \frac{dr}{nu\sin\theta - u} = \frac{rd\theta}{nu\cos\theta} \iff \int_d^r \frac{dr'}{r'} &= \int_0^\theta \frac{nu\sin\theta' - u}{nu\cos\theta'} d\theta \\ \ln(r/d) &= \int_0^\theta \tan\theta' d\theta - \frac{1}{n}\int_0^\theta \sec\theta' d\theta. \end{aligned} \tag{3}$$

Evaluating the remaining integrals, I find

$$\ln(r/d) = \ln(\sec\theta) - \frac{1}{n}\ln\left(\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right) = \ln\left(\frac{\sec\theta}{\left(\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{1/n}}\right), \tag{4}$$

which tells me that the path of the boat satisfies

$$\boxed{r(\theta) = \frac{d\sec\theta}{\left(\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{1/n}}} \tag{5}$$

where d is the width of the river and n is the ratio of the water speed to the boat speed relative to the water.

In Fig.[2] I plot the polar equation (5) for various current speeds. The critical ratio n at which the sailor cannot make it across is determined by $r'(\pi/2) = 0$, or

$$\begin{aligned}
0 &= \lim_{\theta\to\pi/2} \frac{d\sec\theta\tan\theta\left(\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{1/n} - \frac{d}{2n}\sec\theta\left(\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{1/n-1}\sec^2\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}{\left(\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{2/n}} \\
&= \lim_{\theta\to\pi/2} \sec\theta\left(\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{-1/n-1}\left[2n\tan\theta\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) - \sec^2\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right] \\
&= \lim_{\theta\to\pi/2} \sec^2\theta\left(\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{-1/n}\left[2n\sin\theta - \cos\theta\frac{1-\tan^2\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}\right] \\
&= \lim_{\theta\to\pi/2} \sec^2\theta\left(\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{-1/n}\left[n\sin\theta - 1\right]
\end{aligned} \tag{6}$$

As $\theta \to \pi/2$, the quantity outside the brackets diverges. We require then that the term in brackets should vanish, i.e. that $n = 1$. Hence, for water speeds $v > u$ the sailor cannot make it across.

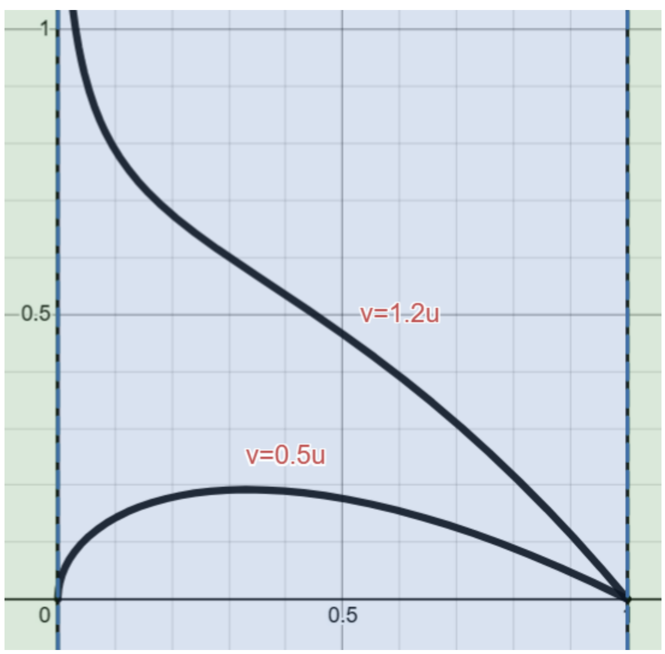

Figure 2: Boat paths for $n = 0.5$ and $n = 1.2$. For $n = 0.5$ the boat is able to make it directly across the canal (from right to left), but for $n = 1.2$ the current sweeps it away.