Simplicies and n-Dimensional Symmetric Functions

Ethan Huecker

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Suppose that $f(t_1, t_2, \cdots, t_n)$ is a symmetric function, then,

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\; f(t_1, t_2, \cdots, t_n) = \frac{1}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n\; f(t_1, t_2, \cdots, t_n). \tag{1}$$

This relation is used in quantum field theory in the derivation of the unitary time-evolution operator $U(t_0, t)$ in the interaction picture, namely the Dyson series representation (time-ordered exponential). The symmetric function $f(t_1, t_2, \cdots, t_n)$ in that context is the time-ordered product of a set of fields.

Proof: I will show the process for $n = 3$ then generalize. Given the integral

$$I_3 = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3\; f(t_1, t_2, t_3), \tag{2}$$

The idea is to perform is many changes of variable as there are integrations. There are 3! here, one of which is trivial,

$$\begin{array}{ll} (t_1 \iff t_1,\ t_2 \iff t_2,\ t_3 \iff t_3) & (t_1 \iff t_1,\ t_2 \iff t_3,\ t_3 \iff t_2) \\ (t_1 \iff t_2,\ t_2 \iff t_3,\ t_3 \iff t_1) & (t_1 \iff t_2,\ t_2 \iff t_1,\ t_3 \iff t_3) \\ (t_1 \iff t_3,\ t_2 \iff t_1,\ t_3 \iff t_2) & (t_1 \iff t_3,\ t_2 \iff t_2,\ t_3 \iff t_1) \end{array} \tag{3}$$

Since the function $f(t_1, t_2, t_3)$ is completely symmetric, each of the 3! substitutions can have their integrands be rewritten for $f(t_1, t_2, t_3)$. The bounds of integration will all be different, but the idea is that each of these substitutions yields a simplex,[1] with the "volume" of each simplex equivalent because they are all related by a substitution.

In this $n = 3$ (3-dim) case, the 3! simplicies are simply tetrahedra of equal volume which combine to form a complete cube of side length $t - t_0$. With this in mind, we can forget about the simplices themselves and simply perform the integration from $t_0 \to t$ for each dimension! This integration is 3! times the volume of a single simplex, which immediately implies the relation

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3\; f(t_1, t_2, t_3) = \frac{1}{3!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_3\; f(t_1, t_2, t_3). \tag{4}$$

It is now very easy to abstract this argument. Given the integral on the l.h.s of (1), the idea is to perform $n!$ changes of variable, where each is an element of the set of permutations on the set $\{t_1, t_2, \cdots, t_n\}$, of which there are $n!$. Each of the $n!$ permutations will form a simplex of an n-cube, with each simplex having equal "volume". With this in mind, we can forgot about the simplices themselves and integrate from $t_0 \to t$ on each dimension of the hypercube. The resulting integral is $n!$ times the contribution to a single simplex, which immediately implies the relation (1). $\square$

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[1] Simplex: The simplest possible polytope in any dimension.