Note on the Lambert W Function

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The Lambert W function is a multi-valued function whose branches are the inverse of the function xe^x,

$$W_k\left(xe^x\right) = x, \qquad k \in \mathbb{Z}. \tag{1}$$

When dealing with real-numbers only, it suffices to the use the principle branch W_0, or even W_{-1} if necessary. For example, $ye^y = x$ can be solved for y iff $x \geq -1/e$, and so $y = W_0(x)$ for $x \geq 0$ and there are two solutions $y = W_0(x), W_{-1}(x)$ for $-1/e \leq x < 0$. There are a few helpful properties of $W(x)$,

$$e^{nW(x)} = \left(\frac{x}{W(x)}\right)^n, \qquad\qquad W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n, \tag{2}$$

where the latter is the MacLaurin series of the principal branch only.

I will showcase the Lambert W function by evaluating the following "power-tower" integral

$$I = \int_0^1 (x^x)^{(x^x)^{\cdots}}\, dx, \tag{3}$$

where the quantity (x^x) is nested infinitely.[1] First, note that the integrand can be expressed as $y = (x^x)^y$, and it follows that

$$\begin{aligned} \ln(y) = y\ln(x^x) \iff -\ln(y)e^{-\ln(y)} &= -x\ln(x) \\ -\ln(y) &= W(-x\ln(x)) \\ y &= e^{-W(-x\ln(x))}. \end{aligned} \tag{4}$$

Appealing to the relation (2), I find that

$$y = -\frac{W(-x\ln(x))}{\ln(x)}. \tag{5}$$

The integral (3) now becomes

$$I = -\int_0^1 \frac{W(-x\ln(x))}{\ln(x)}\, dx = -\int_0^1 \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \frac{(-x\ln(x))^n}{x\ln(x)}\, dx = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \int_0^1 x^{n-1} \ln(x)^{n-1}\, dx, \tag{6}$$

where I used the MacLaurin series (2) and assumed the series is uniformly convergent. From my intuition I see an appearance of the gamma function $\Gamma(x)$, and so I will make the substitution $v = -n\ln(x)$ such that

$$I = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \frac{(-1)^{n-1}}{n} \int_0^{\infty} \left(\frac{v}{n}\right)^{n-1} e^{-v}\, dv = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{(n-1)!}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \eta(2). \tag{7}$$

Since $\eta(2) = \pi^2/12$, I conclude that

$$\boxed{\int_0^1 (x^x)^{(x^x)^{\cdots}}\, dx = \frac{\pi^2}{12}} \tag{8}$$

which is quite a lovely result.

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[1] It can be shown that the same integral but with the pairing (x) diverges.