Summing Infinite Series via Residues

Ethan Huecker

Aug 2023

Preface: The reference used for this note is Zill's text on complex analysis [1].

I will show that certain infinite series can be computed from the formula

$$\sum_{k=-\infty}^{\infty} \frac{1}{p(k)} = -\sum_{j=1}^{r} \text{Res}\left(\frac{\pi\cot(\pi z)}{p(z)}, z_{p_j}\right), \tag{1}$$

where $\{z_{p_j}\}$ is the set of all r-zeros of the degree-r polynomial $p(z)$. For this to hold, $p(z)$ must have real coefficients, degree $r \geq 2$, and no integer zeros.

Proof: Define $f(z) = \pi\cot(\pi z)/p(z)$, and recognize that $\cot(\pi z)$ has poles $\forall z \in \mathbb{Z}$. Furthermore, take the following contour in the complex plane:

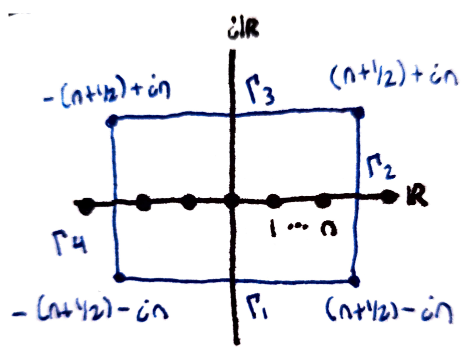

Figure 1: Rectangular contour in the complex plane.

Eventually I will take $n \to \infty$ to capture all of the poles of $\cot(\pi z)$, which will naturally include all of the zeros of $p(z)$. By the Cauchy Residue Theorem, I have

$$\oint_C \frac{\pi\cot(\pi z)}{p(z)} dz = 2\pi i \left(\sum_{k=-n}^{n} \text{Res}\left(\frac{\pi\cot(\pi z)}{p(z)}, k\right) + \sum_{j=1}^{r} \text{Res}\left(\frac{\pi\cot(\pi z)}{p(z)}, z_{p_j}\right)\right), \tag{2}$$

as the contour integral of $f(z)$ along $C = \bigcup_i \Gamma_i$. The poles of $\cot(\pi z)$ are simple poles (order-1), so the residue is simple to compute:

$$\begin{aligned}\text{Res}\left(\frac{\pi\cot(\pi z)}{p(z)}, k\right) &= \lim_{z\to k} \frac{(z-k)\pi\cos(\pi z)}{\sin(\pi z)p(z)} \\ &= \lim_{z\to k} \frac{\pi\cos(\pi z) - (z-\pi)\pi^2\sin(\pi z)}{\pi\cos(\pi z)p(z) + \sin(\pi z)p'(z)} \\ &= \frac{1}{p(k)}.\end{aligned} \tag{3}$$

Therefore, taking the limit as $n \to \infty$ of (2) we have

$$\lim_{n\to\infty} \oint_C \frac{\pi\cot(\pi z)}{p(z)} dz = 2\pi i \left(\sum_{k=-\infty}^{\infty} \frac{1}{p(k)} + \sum_{j=1}^{r} \text{Res}\left(\frac{\pi\cot(\pi z)}{p(z)}, z_{p_j}\right)\right). \tag{4}$$

In order to determine the l.h.s, I will inspect the behavior of $\cot(\pi z)$ on each contour. On Γ_3, parameterize the path as $z = x + in$, where $x \in [-(n+1/2), (n+1/2)]$. Then,

$$\begin{aligned} |\cot(\pi z)| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| &= \left| \frac{e^{i\pi x - \pi n} + e^{-i\pi x + \pi n}}{e^{i\pi x - \pi n} - e^{-i\pi x + \pi n}} \right| \\ &\leq \frac{|e^{i\pi x - \pi n}| + |e^{-i\pi x + \pi n}|}{|e^{-i\pi x + \pi n}| - |e^{i\pi x - \pi n}|} \\ &= \frac{e^{-\pi n} + e^{\pi n}}{e^{\pi n} - e^{-\pi n}} \\ &= \coth(\pi n). \end{aligned} \tag{5}$$

Which is good, because as $n \to \infty$, $\coth(\pi n) \to 1$. Note that on Γ_1 we have the parameterization $z = x - in$, which will result in the same condition. On Γ_4, parameterize the path by $z = -(n+1/2) + iy$, where $y \in [-n, n]$. Then,

$$\begin{aligned} |\cot(\pi z)| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| &= \left| \frac{e^{-i\pi(n+1/2) - \pi y} + e^{i\pi(n+1/2) + \pi y}}{e^{-i\pi(n+1/2) - \pi y} - e^{i\pi(n+1/2) + \pi y}} \right| \\ &\leq \frac{|e^{-i\pi(n+1/2) - \pi y}| + |e^{i\pi(n+1/2) + \pi y}|}{|e^{i\pi(n+1/2) + \pi y}| - |e^{-i\pi(n+1/2) - \pi y}|} \\ &= \frac{e^{-\pi y} + e^{\pi y}}{e^{\pi y} - e^{-\pi y}} \\ &= \coth(\pi y). \end{aligned} \tag{6}$$

Which is also good, as in the $n \to \infty$ limit, $|\coth(\pi y)| < 1$. Clearly, on Γ_2 this argument is still valid, and $|\cot(\pi z)| < 1$.

Therefore, $\cot(\pi z)$ is a bounded function on the entire contour C in the $n \to \infty$ limit. Furthermore, it must hold that

$$\begin{aligned} \left| \oint_C \frac{\pi \cot(\pi z)}{p(z)} dz \right| &\leq \pi \oint_C |\cot(\pi z)| \left| \frac{1}{p(z)} \right| dz \\ &= \pi M \oint_C \frac{dz}{|z|^k} \qquad (M\ const.) \\ &\leq \frac{\pi M}{n^k} \oint_C dz \\ &= \frac{\pi M}{n^k} (8n+1) \xrightarrow{n \to \infty} 0. \end{aligned} \tag{7}$$

So the contour integral evaluates to zero in the $n \to \infty$ limit, because $\cot(\pi z)$ is bounded and the order of $p(z)$ is ≥ 2.[1] Therefore, the l.h.s of (4) vanishes, meaning the term in parentheses vanishes, giving the desired result.

$$\boxed{\sum_{k=-\infty}^{\infty} \frac{1}{p(k)} = -\sum_{j=1}^{r} \text{Res}\left(\frac{\pi \cot(\pi z)}{p(z)}, z_{p_j} \right)} \quad \square \tag{8}$$

[1] This is the same type of argument as Jordan's Lemma.

Example: The Basel Problem

Here, I will use (8) to confirm the solution to the Basel Problem[2]

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{9}$$

Using $p(z) = a^2 + n^2$, then

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2} &= -\sum_{j\in\{-ia,ia\}} \operatorname{Res}\left(\frac{\pi\cot(\pi z)}{a^2+z^2}, j\right) \\ &= -\left[\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{a^2+z^2}, -ia\right) + \operatorname{Res}\left(\frac{\pi\cot(\pi z)}{a^2+z^2}, ia\right)\right] \end{aligned} \tag{10}$$

Factoring the denominator means these poles are only simple poles, so

$$\begin{aligned} \operatorname{Res}\left(\frac{\pi\cot(\pi z)}{(z-ia)(z+ia)}, \pm ia\right) &= \lim_{z\to\pm ia}(z\mp ia)\frac{\pi\cot(\pi z)}{(z-ia)(z+ia)} \\ &= \frac{\pi}{2ia}\cot(i\pi a) \\ &= -\frac{\pi}{2a}\coth(\pi a), \end{aligned} \tag{11}$$

where I used the fact that $\cot(ix) = -i\coth(x)$. Therefore,

$$\begin{aligned} \frac{\pi}{a}\coth(\pi a) &= \sum_{n=-\infty}^{\infty}\frac{1}{a^2+n^2} \\ &= \sum_{n=-\infty}^{-1}\frac{1}{a^2+n^2} + \frac{1}{a^2} + \sum_{n=1}^{\infty}\frac{1}{a^2+n^2} \\ &= \frac{1}{a^2} + 2\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}, \end{aligned} \tag{12}$$

where I used the fact that the summand is an even function. Continuing with a limit $a \to 0$,

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{1}{2}\lim_{a\to 0}\left(\frac{\pi}{a}\coth(\pi a) - \frac{1}{a^2}\right) \\ &= \frac{1}{2}\lim_{a\to 0}\left(\frac{\pi}{a}\left(\frac{1}{\pi a} + \frac{\pi a}{3} + \mathcal{O}(a^2)\right) - \frac{1}{a^2}\right) \\ &= \frac{1}{2}\lim_{a\to 0}\left(\frac{\pi^2}{3} + \mathcal{O}(a)\right) \end{aligned} \tag{13}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}} \qquad \square \tag{14}$$

where I used the Laurent series of $\coth(\pi a)$ to avoid a tough limit.

References

[1] Dennis G. Zill and Patrick D. Shanahan. *Complex Analysis: A First Course with Applications*. Jones & Bartlett Learning, 3 edition, 2013.

[2] This is also $\zeta(2)$, where $\zeta(x)$ is the Riemann zeta function.