Asymptotic Behavior of the Glasser Function - Stationary Phase Analysis

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The Glasser function $G(x)$ is defined by

$$G(x) = \int_0^x dt\, \sin(t \sin t). \tag{1}$$

According to [2], this function has the asymptotic behavior

$$G(x) \sim 2\sqrt{\frac{x}{\pi}}. \tag{2}$$

The original source [1] is (to the best of my knowledge) not online, and there is no validation for this asymptotic behavior online. In this short paper, I will prove (2) using the stationary-phase approximation.

The first set of tricks will be to express the outer sine as a complex exponential, and to partition the integration bounds into regions where the inner sine has a local max or min,

$$G(x) = \mathrm{Im} \int_0^x dt\, e^{it\sin t} = \mathrm{Im} \sum_{n=0}^{\frac{x}{\pi}-1} \int_{n\pi}^{(n+1)\pi} dt\, e^{it\sin t}. \tag{3}$$

The factor of $n\pi$ can be brought into the integrand via the shift of variable $t = n\pi + u$,

$$G(x) = \mathrm{Im} \sum_{n=0}^{\frac{x}{\pi}-1} \int_0^\pi du\, e^{i(n\pi+u)\sin(n\pi+u)} = \mathrm{Im} \sum_{n=0}^{\frac{x}{\pi}-1} \int_0^\pi du\, e^{i(n\pi+u)(-1)^n \sin u} \tag{4}$$

$$= \mathrm{Im} \sum_{n=0}^{\frac{x}{\pi}-1} (-1)^n \int_0^\pi du\, e^{iu\sin u} e^{in(\pi \sin u)}, \tag{5}$$

where the $(-1)^n$ was pulled outside of the exponential in the presence of the Im. For large values of x, or equivalently n, the function $\exp(in(\pi\sin u))$ wildly oscillates. If I define $f(u) = \pi \sin u$, the oscillation rate $nf'(u) = n\pi\cos u$ is of order n. Away from $u_0 = \pi/2$, the phase winds multiple times around the unit circle, and those contributions mutually cancel. Hence, the integrand is dominated by the point $u_0 = \pi/2$, and it suffices to expand the argument of the exponential around u_0

$$G(x) \sim \mathrm{Im} \sum_{n=0}^{\frac{x}{\pi}-1} (-1)^n \int_0^\pi du\, \exp\left(i\pi/2 + in\pi\left(1 - \frac{1}{2}(u-\pi/2)^2\right)\right) \sim \mathrm{Re} \sum_{n=0}^{\frac{x}{\pi}-1} \int_0^\pi du\, \exp\left(-\frac{in\pi}{2}(u-\pi/2)^2\right), \tag{6}$$

where I noted $u\sin u \approx \pi/2$. For large n, the remaining integral is so strongly peaked at $u = \pi/2$ that the integration domain can be safely taken to all of $\mathbb{R}$. The fundamental Gaussian result

$$\int_{\mathbb{R}} du\, \exp\left(-\frac{in\pi}{2}(u-\pi/2)^2\right) = \sqrt{\frac{2\pi}{in\pi}} = \sqrt{\frac{1}{n}}(1-i) \tag{7}$$

applied to (6) results in

$$G(x) \sim \pi + \sum_{n=1}^{\frac{x}{\pi}-1} \frac{1}{\sqrt{n}}, \tag{8}$$

where I was careful to isolate the $n = 0$ term prior to performing the integral. Since $n^{-1/2}$ varies slowly for large n, the sum (8) is well-approximated to leading order in x/π by the integral

$$G(x) \sim \pi + \int_1^{x/\pi - 1} \frac{dn}{\sqrt{n}} \sim 2\sqrt{\frac{x}{\pi}} + O(1), \quad \square \tag{9}$$

and the asymptotic expansion (2) is validated. A plot of $G(x)$ alongside the asymptotic formula (2) is shown in Fig.1.

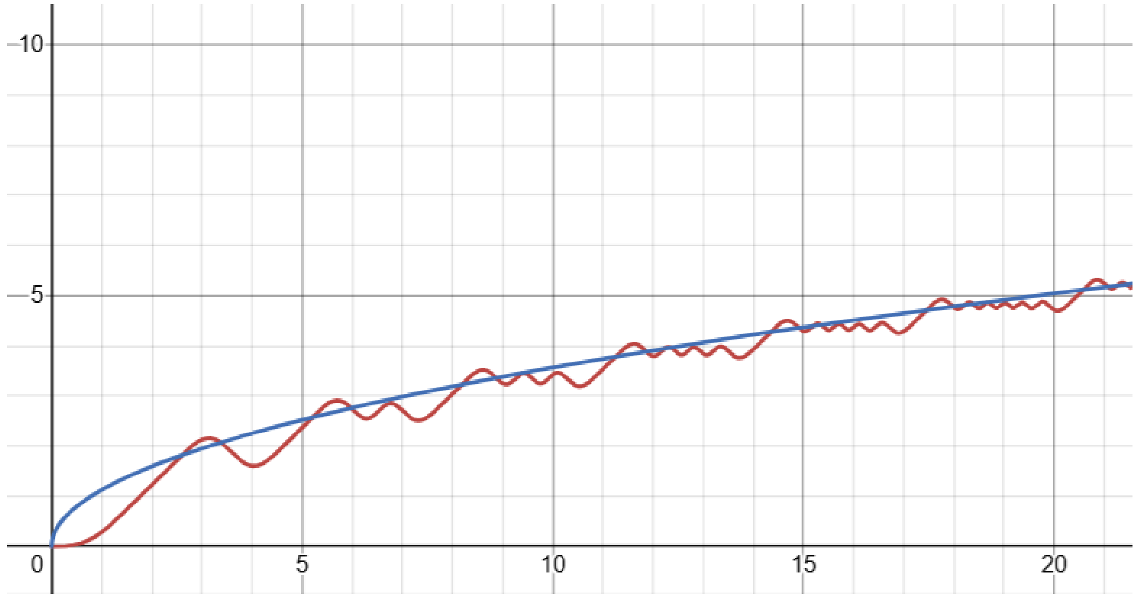

Figure 1: The Glasser function $G(x)$ (1) shown in red alongside the asymptotic form (2) shown in blue.

References

[1] M. L. Glasser and C. Cosgrove. Problem 785. *Nieuw Archief voor Wiskunde*, 8:455, 1990.

[2] Eric W. Weisstein. Glasser function. *MathWorld*—A Wolfram Web Resource. Accessed: 2026-07-17.