The Period of a Pendulum (Right Way)

Ethan Huecker

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I will solve the differential equation describing the motion of a pendulum analytically[1], and determine the first-order corrections to the period.

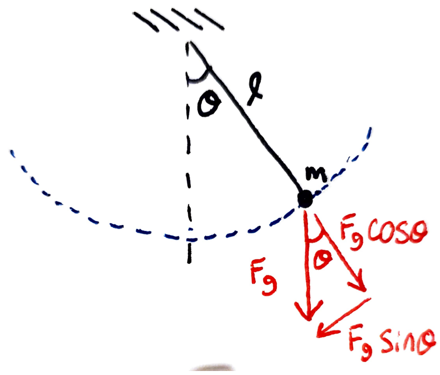

Figure 1: Physicist's Pendulum

Like usual, I will assume the string of length l is massless and rigid, the mass m is point-like, and the only external force is the gravitational force. In polar coordinates, we have that $a = l\ddot{\theta}$, such that Newton's 2nd Law in polar form is

$$ma = -mg\sin\theta \implies \ddot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{1}$$

The trick to solve differential equations like (1) is to express them as the total derivative of some conserved quantity, so I will multiply both sides by $\dot{\theta}$:

$$\begin{aligned} \dot{\theta}\ddot{\theta} + \frac{g}{l}\sin\theta\dot{\theta} = 0 \iff \frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta\right) = 0 \\ \therefore\ \dot{\theta}^2 - \frac{2g}{l}\cos\theta = C \in \mathbb{R}. \end{aligned} \tag{4.02}$$

This is related to the Hamiltonian of the system, i.e. total energy is conserved. Now, I will assume the experimenter starts the pendulum at an initial angle $\theta(0) = \theta_0$ with zero initial speed $\dot{\theta}(0) = 0$. Therefore, at $t = 0$ I find

$$-\frac{2g}{l}\cos\theta_0 = C, \tag{2}$$

which fixes the constant. We then have

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0) \equiv \frac{\alpha}{2}(\cos\theta - \cos\theta_0), \tag{3}$$

where I defined $\alpha = 4g/l$ for future convenience. Since $\cos\theta = 1 - 2\sin^2(\theta/2)$, I can further simplify this as

$$\dot{\theta}^2 = \alpha\left(\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right) \equiv \alpha\left(\beta^2 - \sin^2\left(\frac{\theta}{2}\right)\right), \tag{4}$$

where I defined $\beta = \sin(\theta_0/2)$. Note that the time required for the pendulum to go from angle of zero to an angle of θ_0 is $\tau/4$, where τ is the full period of motion. I will choose these to be my integral bounds, as I want an expression for the period τ as a function of initial angle θ_0. Taking a square-root and integrating both sides of (4) gives

$$\int_0^{\theta_0} \frac{d\theta}{\sqrt{\beta^2 - \sin^2(\theta/2)}} = \sqrt{\alpha}\int_0^{\tau/4} dt = \frac{\tau\sqrt{\alpha}}{4}. \tag{5}$$

[1] If you consider an infinite series to be an analytical solution.

The l.h.s of (5) is reminiscent of an elliptic integral, but needs some manipulations. I will do the substitution $\sin(\theta/2) = \beta \sin\phi$, which implies that

$$\frac{1}{2}\cos(\theta/2)d\theta = \beta\cos\phi\, d\phi \implies d\theta = \frac{2\beta\cos\phi\, d\phi}{\sqrt{1-\sin^2(\theta/2)}} = \frac{2\beta\cos\phi\, d\phi}{\sqrt{1-\beta^2\sin^2\phi}}. \tag{6}$$

and the integration domain changes to $\phi \in [0, \pi/2]$. This gives

$$\begin{aligned}\frac{\tau\sqrt{\alpha}}{4} = \int_0^{\pi/2} \frac{2\beta\cos\phi\, d\phi}{\sqrt{1-\beta^2\sin^2\phi}\sqrt{\beta^2-\beta^2\sin^2\phi}} &= 2\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\beta^2\sin^2\phi}} \\ &= 2K[\beta],\end{aligned} \tag{7}$$

where $K[\beta]$ is the complete elliptic integral of the first kind. Solving for the period τ and back-substituting the values of α, β gives

$$\boxed{\tau = 4\sqrt{\frac{l}{g}}K\left[\sin\left(\frac{\theta_0}{2}\right)\right]} \tag{8}$$

as the exact period of the simple pendulum. Now, $K[k]$ can be written as the infinite sum

$$K[k] = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}k^n\right)^2, \tag{9}$$

where $n!!$ is the "double factorial" defined as the product of the first n integers with the same parity (odd or even) as n. I show this in Appendix A. As a physicist, I only really care about the first 2 terms in the sum,

$$\begin{aligned}K\left[\sin\left(\frac{\theta_0}{2}\right)\right] &\approx \frac{\pi}{2}\left(1+\left(\frac{1}{2}\right)^2\sin^2\left(\frac{\theta_0}{2}\right)\right) \\ &\approx \frac{\pi}{2}\left(1+\frac{1}{16}\theta_0^2\right),\end{aligned} \tag{10}$$

Therefore, (8) can be written as

$$\boxed{\tau \approx 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}\theta_0^2\right)} \tag{11}$$

for the period of a simple pendulum up to first order in θ_0.

A Series Representation of $K[\beta]$

In this section I will derive the series representation of the complete elliptical integral of the first-kind

$$K(k) = \int_0^{\pi/2} \frac{dv}{\sqrt{1 - k^2 \sin^2 v}}. \tag{12}$$

To do so, I need to write the integrand as a power series, so I will determine the n-th derivative of $f(x) = (1-x)^{-1/2}$:

$$\begin{aligned} f^{(1)}(x) &= \frac{1}{2}(1-x)^{-3/2} \\ f^{(2)}(x) &= \frac{1}{2}\frac{3}{2}(1-x)^{-5/2} \\ &\dots \\ f^{(n)}(x) &= \frac{(2n-1)!!}{2^n}(1-x)^{-(2n+1)/2} \implies f^{(n)}(0) = \frac{(2n-1)!!}{2^n}. \end{aligned} \tag{13}$$

Here, I used the definition of the "double-factorial" $n!!$, which is the product of the integers from 1 to n with the same parity (odd, even) as n. Therefore the series representation of $(1-x)^{-1/2}$ has the form

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!2^n} x^n \implies \frac{1}{\sqrt{1-k^2\sin^2 v}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!2^n} k^{2n} \sin^{2n} v, \tag{14}$$

implying that (12) can be rewritten as

$$K(k) = \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!2^n} k^{2n} \sin^{2n} v \, dv = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!2^n} k^{2n} \int_0^{\pi/2} \sin^{2n} v \, dv, \tag{15}$$

where I swapped the integral and sum because I am a physicist.[2] Note that this sum is valid for $n = 0$ because $(-1)!! = 1$.

Now, in order to complete the remaining integral I will appeal to the Beta function $B(z_1, z_2)$ defined by

$$\begin{aligned} B(z_1, z_2) &= 2\int_0^{\pi/2} \sin^{2z_1-1}\theta \cos^{2z_2-1}\theta \, d\theta \\ \implies \frac{1}{2} B\left(n + \frac{1}{2}, n\right) &= \int_0^{\pi/2} \sin^{2n} v \, dv, \end{aligned} \tag{16}$$

such that

$$K(k) = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!2^n} \frac{k^{2n}}{2} B\left(n + \frac{1}{2}, n\right). \tag{17}$$

Furthermore, the most important property of the Beta function is its representation in terms of Gamma functions:

$$\begin{aligned} B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} \implies B\left(n+\frac{1}{2}, n\right) &= \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)} \\ &= \left(\frac{(2n-1)!!\sqrt{\pi}}{2^n}\right)\left(\frac{\sqrt{\pi}}{n!}\right), \end{aligned} \tag{18}$$

where I used the facts that

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!\sqrt{\pi}}{2^n}, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma(n+1) = n!. \tag{19}$$

[2] Or, the sum converges uniformly.

So, (17) can be written as

$$K(k) = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!2^n} \frac{k^{2n}}{2} \frac{\pi(2n-1)!!}{n!2^n} = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{n!2^n} \right)^2 k^{2n}, \tag{20}$$

and to finish things off I will use the fact that $n!2^n = (2n)!!$, giving

$$\boxed{K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} k^n \right)^2} \tag{21}$$

which is quoted in (9) of the main text.