Brief Note on Fractional Calculus

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Preface: A reference used for this note is [1].

The essence of fractional calculus is what happens when a non-integer derivative or integral is taken on some function. For what follows, I will define the quantity

$$ {}_cD_x^{-v} f(x) \tag{1}$$

as the fractional integral from $c \to x$ of a function $f(x)$ to an arbitrary order v. In general, the operator $D^{\pm v}$ is called the "differintegral", where the positive/negative superscripts are derivatives/integrals. In fractional calculus we have the extension $v \in \mathbb{R}^+$ instead of $v \in \mathbb{Z}^+$ for regular calculus. It is necessary that the differintegral operator satisfies the property $D^a D^b = D^{a+b}$, which can be interpretted as meaning "two-half integrals is the same as a single integral."

The derivation of the integral form of the differintegral is trivial if you recall Cauchy's formula for repeated integration:

$$ I^v f(x) = \frac{1}{(v-1)!} \int_c^x (x-t)^{v-1} f(t)\, dt, \tag{2}$$

which is clearly under the restriction $v \in \mathbb{Z}^+$ so that factorials of negative numbers are ignored. Though, if the factorial is replaced with the gamma function, $(v-1)! = \Gamma(v)$, we are allowed to interpret $v \in \mathbb{R}^+$ such that we are left with

$$ \boxed{{}_cD_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)\, dt} \tag{3}$$

This is the form of the fractional integral from $c \to x$ of some $f(x)$, and is known as the Riemann-Liouville definition of the fractional integral.[1] In this form it is apparent why I chose to define $v \in \mathbb{R}^+$ and not all of $\mathbb{R}$, as $\Gamma(v)$ is discontinuous for $v \in \mathbb{Z}^-$. In the case where we consider derivatives $-v \to v$ we need an entirely different formula that bypasses this.

As an example, I will show that two half-integrals of the number 1 gives the expected answer of x. Using (3), we have

$$\begin{aligned} {}_0D_x^{-1/2}(1) &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^x (x-t)^{1/2-1}\ dt \\ &= \frac{1}{\sqrt{\pi}} \int_0^x \frac{dt}{\sqrt{x-t}} \\ &= \frac{1}{\sqrt{\pi}} \Big[-2\sqrt{x-t} \Big]_0^x \\ &= 2\sqrt{\frac{x}{\pi}}, \end{aligned} \tag{4}$$

where I used $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The single $\sqrt{x}$ is a nice confirmation of the concept, and hopefully the additional constants will drop for the next round:

$$\begin{aligned} {}_0D_x^{-1/2}\left({}_0D_x^{-1/2}(1)\right) = {}_0D_x^{-1/2}\left(2\sqrt{\frac{x}{\pi}}\right) = \frac{1}{\sqrt{\pi}}\frac{2}{\sqrt{\pi}} \int_0^x (x-t)^{1/2-1}\sqrt{t}\, dt &= \frac{2}{\pi}\int_0^x \frac{dt}{\sqrt{\frac{x}{t}-1}} \\ &= \frac{4x}{\pi}\int_0^{\pi/2} \cos^2\theta\, d\theta \\ &= \frac{4x}{\pi}\left(\frac{\pi}{4}\right). \end{aligned} \tag{5}$$

[1]There are many different definitions of the fractional integral, but this is the most common.

$$\therefore\ {}_0D_x^{-1/2}\left({}_0D_x^{-1/2}(1)\right) = {}_0D_x^{-1/2}\left(2\sqrt{\frac{x}{\pi}}\right) = x. \tag{6}$$

This is a nice proof of concept, and implies (4) truly represents the "half-integral" of the number 1 ... whatever that means.

It is straight-forward to derive the Riemann-Liouville definition of the fractional derivative from (3). Suppose that $v = n - u$, where $0 < v < 1$ and n is the smallest integer greater than u, then

$$\boxed{D_x^u f(x) = \frac{d^n}{dx^n}\left({}_0D_x^{-v} f(x)\right) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(v)}\int_0^x (x-t)^{v-1} f(t)\, dt\right)} \tag{7}$$

is the fractional derivative of order u. For example, if you want $D_x^{1/2} f(x)$, you first compute the fractional half-integral ${}_0D_x^{-1/2}$ then apply the normal integer derivative,

$$D_x^{1/2} f(x) = D_x^1\left({}_0D_x^{-1/2}\right) = \frac{d}{dx}\left({}_0D_x^{-1/2}\right), \tag{8}$$

because the differintegral satisfies the property $D^a D^b = D^{a+b}$. We've already computed the half-integral of 1 in (4), so it is straightforward to find the half-derivative of 1:

$$\begin{aligned} D_x^{1/2}(1) = \frac{d}{dx}\left({}_0D_x^{-1/2}\right) &= \frac{d}{dx}\left(2\sqrt{\frac{x}{\pi}}\right) \\ &= \frac{1}{\sqrt{\pi x}}. \end{aligned} \tag{9}$$

So it doesn't vanish, but rather picks up a variable of order $-\frac{1}{2}$.

Aside: The normal derivative operator is local, but the fractional derivative operator (7) is non-local, meaning that small variations of $f(x)$ far from $D_x^u f(x)\Big|_{x=a}$ affect the derivative. Perhaps fractional calculus could shed light on non-locality in quantum mechanics?

References

[1] Joseph M. Kimeu. Fractional calculus: Definitions and applications. Master's thesis, Western Kentucky University, April 2009. Department of Mathematics and Computer Science.