The Caustic of a Circle

Ethan Huecker

Jun 2023

Define a family of curves $\{C_t\} \subset \mathbb{R}^2$ parameterized by the variable t, which are each solutions to some equation $F(t,x,y) = 0$. Then the envelope $\mathcal{D}$ of the family $\{C_t\}$ is defined as the set

$$\mathcal{D} = \left\{(x,y) \mid x,y,t \in \mathbb{R};\ F(t,x,y) = 0;\ \frac{\partial F}{\partial t}(t,x,y) = 0\right\} \tag{1}$$

The condition $\frac{\partial F}{\partial t}(t,x,y) = 0$ implies that the envelope $\mathcal{D}$ is a unique curve in $\mathbb{R}^2$ that is tangent to all members of $\{C_t\}$. Now, the caustic is defined (in ray optics) to be the envelope of light rays which have been reflected from a curved surface. Each light ray is tangent to this curve, and a boundary of more concentrated light develops that is distinguishable from the background. An example of this effect is the "heart shape" of light formed at the bottom of an empty tea cup. My goal is to mathematically derive this "heart shape", assuming that the base of the cup is perfectly circular. This particular caustic was first derived by Johann Bernoulli, the lesser known Bernoulli, and I will follow his derivation closely.

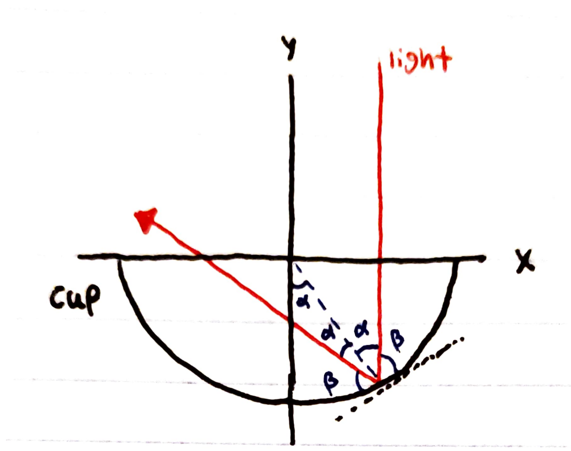

Figure 1: Ray of light reflected off a circle. The fundamental angle relation $\theta_I = \theta_R$ and alternate interior angles implies that each ray of light can be parameterized completely by the angle α defined.

We need the equation of a general reflected ray of parameter α. The point of contact is given simply by $p = (\sin\alpha, -\cos\theta)$, and the slope m of the reflected ray is

$$m = -\frac{\cos(2\alpha)}{\sin(2\alpha)} = -\cot(2\alpha), \tag{2}$$

where I used the fact that total the angle between the incident and reflected ray is 2α. Therefore, I can use the point slope formula to determine the equation of the ray,

$$\begin{aligned} y(x,\alpha) &= -\cot(2\alpha)(x - \sin\alpha) - \cos\alpha \\ &= -\cot(2\alpha)x + \sin\alpha\cot(2\alpha) - \cos\alpha. \end{aligned} \tag{3}$$

To simplify this expression, note that

$$\begin{aligned} \sin\alpha\cot(2\alpha) - \cos\alpha = \frac{1}{2}\sin\alpha(\cot\alpha - \tan\alpha) - \cos\alpha &= -\frac{1}{2}\cos\alpha - \frac{\sin^2\alpha}{2\cos\alpha} \\ &= -\frac{1}{2\cos\alpha}(\sin^2\alpha + \cos^2\alpha) \\ &= -\frac{1}{2}\sec\alpha, \end{aligned} \tag{4}$$

such that the general equation for a reflected light ray is

$$y(x,\alpha) = -\frac{1}{2}\sec\alpha - \cot(2\alpha)x. \tag{5}$$

Now, the caustic must satisfy the condition $\frac{\partial y}{\partial \alpha}(x,\alpha) = 0$, or equivalently

$$0 = \frac{\partial y}{\partial \alpha}(x,\alpha) = -\frac{1}{2}\sec\alpha\tan\alpha + 2\csc^2(2\alpha)x, \tag{6}$$

which naturally defines $x = x(\alpha)$ to be

$$x(\alpha) = \frac{\sec\alpha\tan\alpha}{4\csc^2(2\alpha)} = \frac{1}{4}4\sin^2\alpha\cos^2\alpha\frac{\sin\alpha}{\cos\alpha} = \sin^3\alpha. \tag{7}$$

Substituting $x(\alpha)$ into (5) results in $y(\alpha)$, such that the caustic C is defined parametrically as the set

$$\boxed{C = \left\{(x,y) \mid \alpha \in [0,2\pi),\ x = \sin^3\alpha,\ y = -\frac{1}{2}\sec\alpha - \cot(2\alpha)\sin^3\alpha\right\}} \tag{8}$$

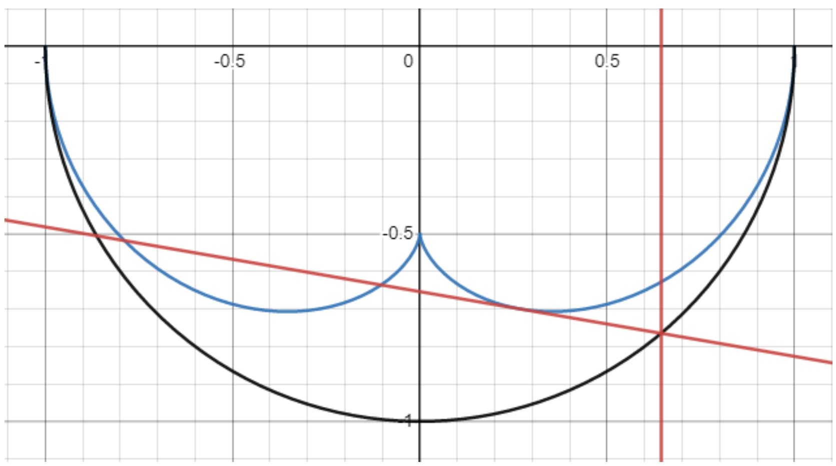

Figure 2: Graph of the caustic of the circle C defined in (8), in blue. The incident and reflected light rays are in red, and the reflected ray is always tangent to C.