

# Solved Series, Products, Limits

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## Series

### Series - 1

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \zeta(2n)}{\pi^{2n}} = \frac{1}{e^2 - 1} \quad (\text{S1.1})$$

**Solution:** Let's rewrite things and using the summation definition for  $\zeta(2n)$ ,

$$\begin{aligned} S &= - \sum_{n=1}^{\infty} \left(-\frac{1}{\pi^2}\right)^n \zeta(2n) \\ &= - \sum_{n=1}^{\infty} \left(-\frac{1}{\pi^2}\right)^n \sum_{m=1}^{\infty} \frac{1}{m^{2n}} \\ &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(-\frac{1}{\pi^2 m^2}\right)^n \\ &= - \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left(-\frac{1}{\pi^2 m^2}\right) \left(-\frac{1}{\pi^2 m^2}\right)^n. \end{aligned} \quad (\text{S1.2})$$

Since the absolute value of the quantity in parentheses is always less than one for all  $m$ , I can evaluate the sum over  $n$  since it is geometric:

$$\begin{aligned} S &= - \sum_{m=1}^{\infty} \frac{-\frac{1}{\pi^2 m^2}}{1 + \frac{1}{\pi^2 m^2}} \\ &= \frac{1}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2 + \frac{1}{\pi^2}}. \end{aligned} \quad (\text{S1.3})$$

To evaluate this I will use the series expansion for  $\coth$ ,

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \frac{1}{m^2 + a^2} &= \frac{\pi}{a} \coth(\pi a) \\ \frac{1}{a^2} + 2 \sum_{m=1}^{\infty} \frac{1}{m^2 + a^2} &= \frac{\pi}{a} \coth(\pi a) \iff \sum_{m=1}^{\infty} \frac{1}{m^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2}, \end{aligned} \quad (\text{S1.4})$$

such that  $S$  really is

$$S = \frac{1}{\pi^2} \left( \frac{\pi^2}{2} \coth(1) - \frac{\pi^2}{2} \right) = \frac{1}{2} \left( \frac{e + e^{-1}}{e - e^{-1}} - 1 \right) = \frac{1}{2} \left( \frac{2e^{-1}}{e - e^{-1}} \right), \quad (\text{S1.5})$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \zeta(2n)}{\pi^{2n}} = \frac{1}{e^2 - 1} \quad (\text{S1.6})$$

## Series - 2

Use Euler's product identity for sine,

$$\sin(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right),$$

as a means of solving the Basel problem.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \quad (\text{S2.1})$$

This is how Euler did it.

**Solution:** First, shift  $x = \pi x$  and expand a little bit,

$$\begin{aligned} \sin(\pi x) &= \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) \\ &= \pi x (1 - x^2) \left(1 - \frac{x^2}{4}\right) \left(1 - \frac{x^2}{9}\right) \left(1 - \frac{x^2}{16}\right) \dots \\ &= \pi x \left(1 - \left(1 + \frac{1}{4}\right)x^2 + \frac{1}{4}x^4\right) \left(1 - \frac{x^2}{9}\right) \left(1 - \frac{x^2}{16}\right) \dots \\ &= \pi x \left(1 - \left(1 + \frac{1}{4} + \frac{1}{9}\right)x^2 + \left(\frac{1}{4} + \frac{5}{36}\right)x^4 - \frac{1}{36}x^6\right) \left(1 - \frac{x^2}{16}\right) \dots \\ &= \pi x \left(1 - \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\right)x^2 + \left(\frac{1}{4} + \frac{5}{36} + \dots\right)x^4 - \left(\frac{1}{36} + \dots\right)x^6 + \mathcal{O}(x^8)\right). \end{aligned} \quad (\text{S2.2})$$

Look at the coefficient in front of  $x^2$ , it is simple to see that if I keep going (to infinity) that it becomes the Basel series. So, to third order I have

$$\sin(\pi x) = \pi x - \pi \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) x^3 + \mathcal{O}(x^5). \quad (\text{S2.3})$$

Let use the Maclaurin expansion of sine to expand the l.h.s to third order,

$$\begin{aligned} \pi x - \frac{\pi^3 x^3}{3!} + \mathcal{O}(x^5) &= \pi x - \pi \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) x^3 + \mathcal{O}(x^5), \\ \iff -\frac{\pi^3}{3!} &= -\pi \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \end{aligned} \quad (\text{S2.4})$$

which solves the Basel problem.

### Series - 3

Derive Lagrange's trigonometric identity

$$\sum_{k=0}^n \cos(k\theta) = \frac{1}{2} \left( 1 + \frac{\sin[(n+1/2)\theta]}{\sin(\theta/2)} \right). \quad (\text{S3.1})$$

**Solution:** Writing the cosine in terms of complex exponentials then using the fact that

$$\sum_{k=0}^n r^k = \frac{r^{n+1} - 1}{r - 1}, \quad (\text{S3.2})$$

for a finite geometric series under  $|r| < 1$  gives

$$\begin{aligned} \sum_{k=0}^n \cos(k\theta) &= \frac{1}{2} \sum_{k=0}^n (e^{ik\theta} + e^{-ik\theta}) = \frac{1}{2} \left( \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} + \frac{e^{-i(n+1)\theta} - 1}{e^{-i\theta} - 1} \right) \\ &= \frac{1}{2} \left( \frac{(e^{i(n+1)\theta} - 1)(e^{-i\theta} - 1) + (e^{-i(n+1)\theta} - 1)(e^{i\theta} - 1)}{(e^{i\theta} - 1)(e^{-i\theta} - 1)} \right) \\ &= \frac{1}{2} \left( \frac{2 - (e^{i\theta} + e^{-i\theta}) + e^{in\theta} + e^{-in\theta} - (e^{i(n+1)\theta} + e^{-i(n+1)\theta})}{2 - (e^{i\theta} + e^{-i\theta})} \right) \\ &= \frac{1}{2} \left( 1 + \frac{e^{in\theta}(1 - e^{i\theta}) + e^{-in\theta}(1 - e^{-i\theta})}{2 - (e^{i\theta} + e^{-i\theta})} \right) \\ &= \frac{1}{2} \left( 1 + \frac{e^{i(n+1/2)\theta}(e^{-i\theta/2} - e^{i\theta/2}) + e^{-i(n+1/2)\theta}(e^{i\theta/2} - e^{-i\theta/2})}{2 - (e^{i\theta} + e^{-i\theta})} \right) \\ &= \frac{1}{2} \left( 1 + \frac{-2ie^{i(n+1/2)\theta} \sin(\theta/2) + 2ie^{-i(n+1/2)\theta} \sin(\theta/2)}{4 \sin^2(\theta/2)} \right) \\ &= \frac{1}{2} \left( 1 + \frac{e^{i(n+1/2)\theta} - e^{-i(n+1/2)\theta}}{2i \sin(\theta/2)} \right) \\ &= \frac{1}{2} \left( 1 + \frac{\sin[(n+1/2)\theta]}{\sin(\theta/2)} \right), \end{aligned} \quad (\text{S3.3})$$

as desired.

## Series - 4

Show the following,

$$\sum_{m=-\infty}^{\infty} e^{-\frac{\alpha}{2}(\theta-2\pi m)^2} = \frac{1}{\sqrt{2\pi\alpha}} \sum_{n=-\infty}^{\infty} e^{in\theta - \frac{n^2}{2\alpha}}, \quad (\text{S4.1})$$

which is used in the Villain approximation in condensed matter physics for the approximation of the partition function term  $\exp(\alpha \cos \theta)$ .

**Solution:** I will perform a F.T.

$$S \equiv \sum_{m=-\infty}^{\infty} e^{-\frac{\alpha}{2}(\theta-2\pi m)^2} = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}, \quad (\text{S4.2})$$

where on the grounds of orthogonality

$$\begin{aligned} b_n &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-in\theta} \sum_{m=-\infty}^{\infty} e^{-\frac{\alpha}{2}(\theta-2\pi m)^2} \\ &= \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-in\theta} e^{-\frac{\alpha}{2}(\theta-2\pi m)^2}. \end{aligned} \quad (\text{S4.3})$$

The trick is to make the substitution  $\phi = \theta - 2\pi m$ , such that

$$\begin{aligned} b_n &= \sum_{m=-\infty}^{\infty} e^{2\pi imn} \int_{-\pi-2\pi m}^{\pi-2\pi m} \frac{d\phi}{2\pi} e^{-in\phi} e^{-\frac{\alpha}{2}\phi^2} \\ &= \sum_{m=-\infty}^{\infty} \int_{-\pi-2\pi m}^{\pi-2\pi m} \frac{d\phi}{2\pi} e^{-in\phi - \frac{\alpha}{2}\phi^2}, \end{aligned} \quad (\text{S4.4})$$

where I used the fact that  $nm \in \mathbb{Z}$ . The summation over  $m$  is now very simple,

$$b_n = \left[ \dots + \int_{\pi}^{3\pi} + \int_{-\pi}^{\pi} + \int_{-3\pi}^{-\pi} + \dots \right] \frac{d\phi}{2\pi} e^{-in\phi - \frac{\alpha}{2}\phi^2} = \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} e^{-in\phi - \frac{\alpha}{2}\phi^2}, \quad (\text{S4.4})$$

and the remaining integral can be evaluated by completing the square,

$$\begin{aligned} b_n &= e^{-\frac{n^2}{2\alpha}} \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} e^{-\frac{\alpha}{2}\phi^2 - in\phi + \frac{n^2}{2\alpha}} = e^{-\frac{n^2}{2\alpha}} \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} e^{-\frac{\alpha}{2}(\phi - \frac{in}{\alpha})^2} \\ &= e^{-\frac{n^2}{2\alpha}} \frac{1}{2\pi} \sqrt{\frac{2\pi}{\alpha}}, \end{aligned} \quad (\text{S4.5})$$

and so (S4.2) can be expressed as

$$\sum_{m=-\infty}^{\infty} e^{-\frac{\alpha}{2}(\theta-2\pi m)^2} = \frac{1}{\sqrt{2\pi\alpha}} \sum_{n=-\infty}^{\infty} e^{in\theta - \frac{n^2}{2\alpha}} \quad (\text{S4.6})$$

as desired.

## Series - 5

$$S = \sum_{k=1}^{\infty} \frac{\zeta(1-2k)}{\Gamma(2k-1)} = \frac{e}{(e-1)^2} - 1 \quad (\text{S5.1})$$

**Solution:** I immediately thought of using the relation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s), \quad (\text{S5.2})$$

and after playing around with it for a bit, I found that it works to use the reflection formula for the gamma function:

$$\Gamma(1-s) = \frac{\pi}{\sin(\pi s) \Gamma(s)} \implies \zeta(s) = 2^s \pi^s \frac{\sin\left(\frac{\pi s}{2}\right) \zeta(1-s)}{\sin(\pi s) \Gamma(s)}. \quad (\text{S5.3})$$

Letting  $s = 2k$  and moving things around,

$$\frac{\zeta(2k)}{2^{2k} \pi^{2k}} \frac{\sin(2\pi k)}{\sin(\pi k)} = \frac{\zeta(1-2k)}{\Gamma(2k)} = \frac{\zeta(1-2k)}{(2k-1)\Gamma(2k-1)} \iff \frac{\zeta(1-2k)}{\Gamma(2k-1)} = \frac{(2k-1) \cos(\pi k)}{2^{2k-1} \pi^{2k}} \zeta(2k), \quad (\text{S5.4})$$

where I used the double-angle identity for the sine. Since  $k \in \mathbb{Z}$ , I can replace  $\cos(\pi k) \rightarrow (-1)^k$  inside the summation:

$$S = \sum_{k=1}^{\infty} \frac{(2k-1)(-1)^k}{2^{2k-1} \pi^{2k}} \zeta(2k) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(2k-1)(-1)^k}{2^{2k-1} \pi^{2k} n^{2k}}. \quad (\text{S5.5})$$

Here's the trick. Note that

$$\frac{2k-1}{n^{2k}} = -\frac{\partial}{\partial n} \left( \frac{1}{n^{2k-1}} \right). \quad (\text{S5.6})$$

Ignoring the question of passing the derivative across the sum (probably allowed by linearity), this gives

$$\begin{aligned} S &= -\sum_{n=1}^{\infty} \frac{\partial}{\partial n} \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k-1} \pi^{2k} n^{2k-1}} = -2 \sum_{n=1}^{\infty} \frac{\partial}{\partial n} \left[ n \sum_{k=1}^{\infty} \left( -\frac{1}{4\pi^2 n^2} \right)^k \right] \\ &= -2 \sum_{n=1}^{\infty} \frac{\partial}{\partial n} \left[ n \left( \sum_{k=0}^{\infty} \left( -\frac{1}{4\pi^2 n^2} \right)^k - 1 \right) \right] \\ &= 2 \sum_{n=1}^{\infty} \frac{\partial}{\partial n} \left[ \frac{n}{1 + 4\pi^2 n^2} \right], \end{aligned} \quad (\text{S5.7})$$

where I manifested the  $k$ -series as geometric, and collected terms. Differentiating, I find

$$S = 2 \sum_{n=1}^{\infty} \frac{1}{1 + 4\pi^2 n^2} - 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2}{(1 + 4\pi^2 n^2)^2}. \quad (\text{S5.8})$$

Looking at a table of series, it appears our friend will be

$$\sum_{n=0}^{\infty} \frac{1}{a^2 + n^2} = \frac{1 + a\pi \coth(a\pi)}{2a^2}, \quad (\text{S5.9})$$

which I have no plan on deriving, yet. Clearly,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{1 + 4\pi^2 n^2} &= \frac{1}{4\pi^2} \left( -(2\pi)^2 + \sum_{n=0}^{\infty} \frac{1}{(1/2\pi)^2 + n^2} \right) = \frac{1}{4\pi^2} \left( -(2\pi)^2 + \frac{(2\pi)^2}{2} \left( 1 + \frac{1}{2} \coth\left(\frac{1}{2}\right) \right) \right) \\ &= \frac{1}{4} \left( \coth\left(\frac{1}{2}\right) - 2 \right). \end{aligned} \quad (\text{S5.10})$$

Such that one of the remaining series can be simplified,

$$S = \frac{1}{2} \left( \coth\left(\frac{1}{2}\right) - 2 \right) - 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2}{(1 + 4\pi^2 n^2)^2}. \quad (\text{S5.11})$$

Now for the other,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^2}{(1+4\pi^2 n^2)^2} &= \frac{1}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{(1/2\pi)^2 + n^2}{((1/2\pi)^2 + n^2)^2} - \frac{1}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{(1/2\pi)^2}{((1/2\pi)^2 + n^2)^2} \\
&= \frac{1}{(2\pi)^4} \left( -(2\pi^2) + \sum_{n=0}^{\infty} \frac{1}{(1/2\pi)^2 + n^2} \right) - \frac{1}{(2\pi)^6} \sum_{n=1}^{\infty} \frac{1}{((1/2\pi)^2 + n^2)^2} \\
&= \frac{1}{4(2\pi)^2} \left( \coth\left(\frac{1}{2}\right) - 2 \right) - \frac{1}{(2\pi)^6} \sum_{n=1}^{\infty} \frac{1}{((1/2\pi)^2 + n^2)^2}
\end{aligned} \tag{S5.12}$$

where I added a zero in the form of  $0 = (1/2\pi)^2 - (1/2\pi)^2$ . Note that by differentiating (S5.9) we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{-2a}{(a^2 + n^2)^2} &= -\frac{1}{a^3} - \frac{\pi}{2} \left( \frac{\coth(\pi a)}{a^2} + \frac{\pi \operatorname{csch}^2(\pi a)}{a} \right) \\
\iff \sum_{n=0}^{\infty} \frac{1}{(a^2 + n^2)^2} &= \frac{1}{2a^4} + \frac{\pi}{4a} \left( \frac{\coth(\pi a)}{a^2} + \frac{\pi \operatorname{csch}^2(\pi a)}{a} \right),
\end{aligned} \tag{S5.13}$$

and so

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{((1/2\pi)^2 + n^2)^2} &= -(2\pi)^4 + \sum_{n=0}^{\infty} \frac{1}{((1/2\pi)^2 + n^2)^2} = -(2\pi)^4 + \frac{(2\pi)^4}{2} + \frac{\pi^2}{2} \left( (2\pi)^2 \coth\left(\frac{1}{2}\right) + 2\pi^2 \operatorname{csch}^2\left(\frac{1}{2}\right) \right) \\
&= -\frac{(2\pi)^4}{2} + \frac{\pi^2(2\pi)^2}{2} \left( \coth\left(\frac{1}{2}\right) + \frac{1}{2} \operatorname{csch}^2\left(\frac{1}{2}\right) \right).
\end{aligned} \tag{S5.14}$$

As a result,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^2}{(1+4\pi^2 n^2)^2} &= \frac{1}{4(2\pi)^2} \left( \coth\left(\frac{1}{2}\right) - 2 \right) + \frac{1}{2(2\pi)^2} - \frac{\pi^2}{2(2\pi)^4} \left( \coth\left(\frac{1}{2}\right) + \frac{1}{2} \operatorname{csch}^2\left(\frac{1}{2}\right) \right) \\
&= \frac{1}{32\pi^2} \left( \coth\left(\frac{1}{2}\right) - \frac{1}{2} \operatorname{csch}^2\left(\frac{1}{2}\right) \right).
\end{aligned} \tag{S5.15}$$

Plugging this into (S5.11), we get

$$S = \frac{1}{2} \left( \coth\left(\frac{1}{2}\right) - 2 \right) - \frac{1}{2} \left( \coth\left(\frac{1}{2}\right) - \frac{1}{2} \operatorname{csch}^2\left(\frac{1}{2}\right) \right) = \frac{1}{4} \operatorname{csch}^2\left(\frac{1}{2}\right) - 1. \tag{S5.16}$$

Since  $\operatorname{csch}^2(1/2) = 4e/(e-1)^2$ , this is simply

$$\sum_{k=1}^{\infty} \frac{\zeta(1-2k)}{\Gamma(2k-1)} = \frac{e}{(e-1)^2} - 1 \tag{S5.17}$$

which is quite nice.

# Products

## Product - 1

$$A = \prod_{n=2}^{\infty} e \left(1 - \frac{1}{n^2}\right)^{n^2} = \frac{\pi}{e\sqrt{e}} \quad (\text{P1.1})$$

**Solution:** The first major trick, which I think is the only way, is to take the natural logarithm of both sides,

$$\begin{aligned} \ln(A) &= \ln \left( \prod_{n=2}^{\infty} e \left(1 - \frac{1}{n^2}\right)^{n^2} \right) = \sum_{n=2}^{\infty} \ln \left( e \left(1 - \frac{1}{n^2}\right)^{n^2} \right) \\ &= \sum_{n=2}^{\infty} 1 + n^2 \ln \left(1 - \frac{1}{n^2}\right), \end{aligned} \quad (\text{P1.2})$$

giving a more manageable infinite sum. Next, I will represent the logarithm as an integral,

$$\ln \left(1 - \frac{1}{n^2}\right) = - \int_0^1 \frac{du}{n^2 - u}, \quad (\text{P1.3})$$

such that (P1.2) becomes

$$\ln(A) = \sum_{n=2}^{\infty} 1 - n^2 \int_0^1 \frac{du}{n^2 - u}. \quad (\text{P1.4})$$

Now, since  $u \in [0, 1] \subset \mathbb{R}$  and  $n \in [2, \infty) \subset \mathbb{N}$ , we have that

$$\int_0^1 du = 1 \quad \text{and} \quad \frac{1}{n^2 - u} > 0, \quad (\text{P1.5})$$

implying that the integral in (P1.4) can be taken outside of everything, including the summation:

$$\ln(A) = \int_0^1 \left( \sum_{n=2}^{\infty} 1 - \frac{n^2}{n^2 - u} \right) du = - \int_0^1 u \left( \sum_{n=2}^{\infty} \frac{1}{n^2 - u} \right) du. \quad (\text{P1.6})$$

The next important trick is to use the following formula,

$$-\frac{\pi}{\sqrt{u}} \cot(\pi\sqrt{u}) = \sum_{n=-\infty}^{\infty} \frac{1}{n^2 - u} = 2 \left( \sum_{n=2}^{\infty} \frac{1}{n^2 - u} \right) - \frac{1}{u} + \frac{2}{1 - u}, \quad (\text{P1.7})$$

to replace the summation in (P1.6), giving

$$\begin{aligned} \ln(A) &= - \int_0^1 u \left( -\frac{\pi}{\sqrt{u}} \cot(\pi\sqrt{u}) + \frac{1}{2u} - \frac{1}{1 - u} \right) du = \frac{1}{2} \int_0^1 \left( \pi\sqrt{u} \cot(\pi\sqrt{u}) - 1 + \frac{2u}{1 - u} \right) du \\ &\xrightarrow{u=v^2} \int_0^1 \left( \pi v^2 \cot(\pi v) + \frac{2v^3}{1 - v^2} - v \right) dv. \end{aligned} \quad (\text{P1.8})$$

For the middle term I will perform polynomial long-division,

$$\frac{2v^3}{1 - v^2} = -2v + \frac{2v}{1 - v^2}, \quad (\text{P1.9})$$

such that (P1.8) becomes

$$\ln(A) = \int_0^1 \left( \pi v^2 \cot(\pi v) - 3v + \frac{2v}{1 - v^2} \right) dv = -\frac{3}{2} + \pi \int_0^1 v^2 \cot(\pi v) dv + \int_0^1 \frac{2v}{1 - v^2} dv. \quad (\text{P1.10})$$

Here is another cool trick. Substitute  $v \rightarrow 1 - v$  in the middle integral:

$$\begin{aligned}
\int_0^1 v^2 \cot(\pi v) dv &\xrightarrow{v=1-v} - \int_1^0 (1-v)^2 \cot(\pi - \pi v) dv \\
&= - \int_0^1 (1-2v+v^2) \cot(\pi v) dv \\
&= - \int_0^1 v^2 \cot(\pi v) dv + 2 \int_0^1 v \cot(\pi v) dv - \int_0^1 \cot(\pi v) dx.
\end{aligned} \tag{P1.11}$$

Here, I can add the original integral to both sides of the equation, then divide by two to give an interesting result

$$\int_0^1 \pi v^2 \cot(\pi v) dv = \int_0^1 \pi v \cot(\pi v) dv, \tag{P1.12}$$

such that (P1.10) becomes

$$\begin{aligned}
\ln(A) &= -\frac{3}{2} + \int_0^1 \pi v \cot(\pi v) dv + \int_0^1 \frac{2v}{1-v^2} dv \\
&= -\frac{3}{2} + \lim_{\lambda \rightarrow 1^-} \int_0^\lambda \pi v \cot(\pi v) dv + \lim_{\lambda \rightarrow 1^-} \int_0^\lambda \frac{2v}{1-v^2} dv,
\end{aligned} \tag{P1.13}$$

where I took limits to account for the discontinuities. It is now time to explicitly evaluate the integrals. First,

$$\begin{aligned}
\lim_{\lambda \rightarrow 1^-} \int_0^\lambda \frac{2v}{1-v^2} dv &= - \lim_{\lambda \rightarrow 1^-} \ln(1-\lambda^2) \\
&= - \lim_{\lambda \rightarrow 1^-} \ln(1+\lambda) - \lim_{\lambda \rightarrow 1^-} \ln(1-\lambda),
\end{aligned} \tag{P1.14}$$

which I expanded for future convenience. Second, by integration by parts we have

$$\begin{aligned}
\lim_{\lambda \rightarrow 1^-} \int_0^\lambda \pi v \cot(\pi v) dv &= \lim_{\lambda \rightarrow 1^-} v \ln(\sin(\pi v)) \Big|_0^\lambda - \int_0^1 \ln(\sin(\pi v)) dv \\
&= \lim_{\lambda \rightarrow 1^-} \lambda \ln(\sin(\pi \lambda)) - \lim_{\alpha \rightarrow 0} \alpha \ln(\sin(\pi \alpha)) - \int_0^1 \ln(\sin(\pi v)) dv \\
&= \lim_{\lambda \rightarrow 1^-} \lambda \ln(\sin(\pi \lambda)) - \int_0^1 \ln(\sin(\pi v)) dv,
\end{aligned} \tag{P1.15}$$

where I noted that the middle limit vanishes. I can now substitute (P1.14) and (P1.15) into (P1.13) to give

$$\begin{aligned}
\ln(A) &= -\frac{3}{2} + \lim_{\lambda \rightarrow 1^-} \left( \lambda \ln(\sin(\pi \lambda)) - \ln(1+\lambda) - \ln(1-\lambda) \right) - \int_0^1 \ln(\sin(\pi v)) dv \\
&= -\frac{3}{2} - \ln(2) + \lim_{\lambda \rightarrow 1^-} \left[ \lambda \ln \left( \frac{\sin(\pi \lambda)}{1-\lambda} \right) \right] - \int_0^1 \ln(\sin(\pi v)) dv.
\end{aligned} \tag{P1.16}$$

using L'Hopital's rule, along with the fact that the natural logarithm is continuous along its domain, implies

$$\lim_{\lambda \rightarrow 1^-} \left[ \lambda \ln \left( \frac{\sin(\pi \lambda)}{1-\lambda} \right) \right] = \ln \left( \lim_{\lambda \rightarrow 1^-} \left[ \frac{\sin(\pi \lambda)}{1-\lambda} \right] \right) = \ln(\pi), \tag{P1.17}$$

such that all that remains of our original problem is

$$\ln(A) = -\frac{3}{2} + \ln(\pi) - \ln(2) - \int_0^1 \ln(\sin(\pi v)) dv. \tag{P1.18}$$

To solve the last integral, I will first let  $v \rightarrow 2v$ :

$$\begin{aligned}
\int_0^1 \ln(\sin(\pi v)) dv &= 2 \int_0^{1/2} \ln(\sin(2\pi v)) dv \\
&= 2 \int_0^{1/2} \ln(2) dv + 2 \int_0^{1/2} \ln(\sin(\pi v)) dv + 2 \int_0^{1/2} \ln(\cos(\pi v)) dv,
\end{aligned} \tag{P1.19}$$

where I used the double angle identity for  $\sin(2\pi v)$  in the second step. Now, recognize that

$$\int_0^{1/2} \ln(\cos(\pi v)) dv = \int_{1/2}^1 \ln(\sin(\pi v)) dv, \quad (\text{P1.20})$$

which allows me to combine the two integrals, giving

$$\int_0^1 \ln(\sin(\pi v)) dv = \ln(2) + 2 \int_0^{1/2} \ln(\sin(\pi v)) dv \implies \int_0^1 \ln(\sin(\pi v)) dv = -\ln(2), \quad (\text{P1.21})$$

such that (P1.18) can be written as

$$\ln(A) = -\frac{3}{2} + \ln(\pi) - \ln(2) + \ln(2) = -\frac{3}{2} + \ln(\pi). \quad (\text{P1.22})$$

To get the final answer, just exponentiate:

$$A = \pi e^{-3/2} = \frac{\pi}{e\sqrt{e}} \implies \prod_{n=2}^{\infty} e \left(1 - \frac{1}{n^2}\right)^{n^2} = \frac{\pi}{e\sqrt{e}} \quad (\text{P1.23})$$

## Product - 2

$$P = \prod_{n=1}^{\infty} \frac{(5n-2)(5n-3)}{(5n-1)(5n-4)} = \phi, \quad (\text{P2.1})$$

Here  $\phi = (1 + \sqrt{5})/2$  is the golden ratio.

**Solution:** To solve this I will use the Euler product representation

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^z}{1 + \frac{z}{n}} \iff \frac{z\Gamma(z)}{5} = \prod_{n=1}^{\infty} \frac{1}{n^{z-1}} \frac{(n+1)^z}{5n+5z}, \quad (\text{P2.2})$$

such that

$$\begin{aligned} \left( -\frac{\Gamma(-\frac{1}{5})}{25} \right) \left( -\frac{25}{2(-\frac{2}{5})} \right) \left( -\frac{25}{3\Gamma(-\frac{3}{5})} \right) \left( -\frac{4\Gamma(-\frac{4}{5})}{25} \right) &= \prod_{n=1}^{\infty} \frac{(5n-2)(5n-3)}{(5n-1)(5n-4)} \\ &\times \frac{(n+1)^{-1/5}}{n^{-6/5}} \frac{n^{-7/5}}{(n+1)^{-2/5}} \frac{n^{-8/5}}{(n+1)^{-3/5}} \frac{(n+1)^{-4/5}}{n^{-9/5}} \\ &= \prod_{n=1}^{\infty} \frac{(5n-2)(5n-3)}{(5n-1)(5n-4)}, \end{aligned} \quad (\text{P2.3})$$

which is quite elegant. Therefore,

$$\begin{aligned} P &= \frac{2\Gamma(-\frac{1}{5})\Gamma(-\frac{4}{5})}{3\Gamma(-\frac{2}{5})\Gamma(-\frac{3}{5})} = \frac{\Gamma(\frac{1}{5})\Gamma(\frac{4}{5})}{\Gamma(\frac{2}{5})\Gamma(\frac{3}{5})} = \frac{\Gamma(\frac{1}{5})\Gamma(1-\frac{1}{5})}{\Gamma(\frac{2}{5})\Gamma(1-\frac{2}{5})} \\ &= \frac{\sin(\frac{2\pi}{5})}{\sin(\frac{\pi}{5})} \\ &= 2\cos\left(\frac{\pi}{5}\right) \\ &= \frac{1+\sqrt{5}}{2}, \quad \square \end{aligned} \quad (\text{P2.4})$$

where the second equality uses  $\Gamma(n+1) = n\Gamma(n)$  and the fourth uses the Euler reflection formula.

# Limits

## Limit - 1

$$L = \lim_{n \rightarrow \infty} \frac{(3n-1)!n^2}{27^n(n!)^3} = \frac{1}{2\pi\sqrt{3}} \quad (\text{L1.1})$$

**Solution:** For large  $n$  we can replace the factorials with the Stirling approximation,

$$n! \rightarrow \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \quad (3n-1)! \rightarrow \sqrt{2\pi(3n-1)} \frac{(3n-1)^{3n-1}}{e^{3n-1}}, \quad (\text{L1.2})$$

giving

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \frac{n^2 (2\pi(3n-1))^{1/2} (3n-1)^{3n-1} e^{3n}}{27^n (2\pi n)^{3/2} e^{3n-1} n^{3n}} \\ &= \frac{e}{2\pi} \lim_{n \rightarrow \infty} \frac{1}{3^{3n}} \frac{(3n-1)^{3n-1/2}}{n^{3n-1/2}} \\ &= \frac{e}{2\pi\sqrt{3}} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n}\right)^{3n-1/2} \\ &= \frac{e}{2\pi\sqrt{3}} \frac{\lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n}\right)^{3n}}{\lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n}\right)^{1/2}} \\ &= \frac{e}{2\pi\sqrt{3}} \lim_{n \rightarrow \infty} \exp\left(3n \ln\left(1 - \frac{1}{3n}\right)\right) \\ &= \frac{e}{2\pi\sqrt{3}} \exp\left(-3 \lim_{n \rightarrow \infty} \frac{1}{3\left(1 - \frac{1}{3n}\right)}\right) \implies L = \frac{1}{2\pi\sqrt{3}} \end{aligned} \quad (\text{L1.3})$$

I dragged the limit across the exponential since it is continuous and used l'Hopital's rule.

## Limit - 2

$$L = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{e^n} \exp \left( \int_0^\infty \left\lfloor \frac{n}{e^x} \right\rfloor dx \right) = \frac{1}{\sqrt{2\pi}} \quad (\text{L2.1})$$

**Solution:** Clearly the integral should simplify. Let's do a substitution of  $u = n/e^x$  such that

$$\begin{aligned} \int_0^\infty \left\lfloor \frac{n}{e^x} \right\rfloor dx &= \int_0^n \frac{\lfloor u \rfloor}{u} du \\ &= \int_0^1 \frac{0}{u} du + \int_1^2 \frac{1}{u} du + \int_2^3 \frac{2}{u} du + \dots + \int_{n-1}^n \frac{n-1}{u} du \\ &= \sum_{j=1}^{n-1} \int_j^{j+1} \frac{j}{u} du \\ &= \sum_{j=1}^{n-1} j(\ln(j+1) - \ln(j)) \\ &= 1(\ln(2) - \ln(1)) + 2(\ln(3) - \ln(2)) + \dots + (n-1)(\ln(n) - \ln(n-1)) \\ &= -\ln(2) - \ln(3) - \dots - \ln(n-1) + (n-1)\ln(n) \\ &= -\ln((n-1)!) + \ln(n^{n-1}) \\ &= \ln \left( \frac{n^{n-1}}{(n-1)!} \right). \end{aligned} \quad (\text{L2.2})$$

In the second equality I used the fact that when  $\lfloor u \rfloor$  is integrated it picks out the lower bound, and I made use of a semi-telescoping series with some trivial logarithm properties. Clearly then, (L2.1) becomes

$$L = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{e^n} \frac{n^{n-1}}{(n-1)!}, \quad (\text{L2.3})$$

which can be simplified by noting that for large  $n$ ,

$$(n-1)! \rightarrow \sqrt{2\pi(n-1)} \left( \frac{n-1}{e} \right)^{n-1}, \quad (\text{L2.4})$$

which is Stirling's approximation. Thus,

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \sqrt{\frac{n}{2\pi(n-1)}} \left( \frac{e^{n-1}}{e^n} \right) \left( \frac{n}{n-1} \right)^{n-1} \\ &= \frac{1}{e\sqrt{2\pi}} \lim_{n \rightarrow \infty} \left( \frac{n}{n-1} \right)^n \\ &= \frac{1}{e\sqrt{2\pi}} \exp \left( \lim_{n \rightarrow \infty} n \ln \left( \frac{n}{n-1} \right) \right) = \frac{1}{e\sqrt{2\pi}} \exp \left( \lim_{n \rightarrow \infty} \left( \frac{n}{n-1} \right) \right) = \frac{1}{\sqrt{2\pi}} \end{aligned} \quad (\text{L2.5})$$

Where in the second equality is used the fact that the limit of a product is a product of the limits, and in the fourth equality I used l'Hopital's rule.

### Limit - 3

$$L = \lim_{x \rightarrow 1^-} \left( \frac{1}{x} - 1 \right) \sum_{n=1}^{\infty} x^n \ln(1 + x^n) = 2 \ln(2) - 1 \quad (\text{L3.1})$$

**Solution:** A neat trick is to express the logarithm as an integral and to expand the integrand in a common Maclaurin series, after which the integral can be performed and l'Hopital's rule can be used:

$$\begin{aligned} L &= \lim_{x \rightarrow 1^-} \left( \frac{1}{x} - 1 \right) \sum_{n=1}^{\infty} x^n \ln(1 + y) \Big|_0^{x^n} = \lim_{x \rightarrow 1^-} \left( \frac{1}{x} - 1 \right) \int_0^{x^n} dy \sum_{n=1}^{\infty} \frac{x^n}{1 + y} \\ &= \lim_{x \rightarrow 1^-} \left( \frac{1}{x} - 1 \right) \int_0^{x^n} dy \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m x^n y^m \\ &= \lim_{x \rightarrow 1^-} \left( \frac{1}{x} - 1 \right) \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{m+1} (x^{m+2})^n \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m+1} \lim_{x \rightarrow 1^-} \frac{(\frac{1}{x} - 1)}{1 - x^{m+2}} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)} \lim_{x \rightarrow 1^-} \frac{-1/x^2}{x^{m+1}} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)} \\ &= 2 \ln(2) - 1, \end{aligned} \quad (\text{L3.2})$$

$$\therefore \lim_{x \rightarrow 1^-} \left( \frac{1}{x} - 1 \right) \sum_{n=1}^{\infty} x^n \ln(1 + x^n) = 2 \ln(2) - 1 \quad (\text{L3.3})$$

which is quite an elegant result.

### Limit - 4

$$L = \lim_{x \rightarrow 0} \left( \frac{x^\pi - \pi^x}{x^e - e^x} \right)^{\csc(x)} = \frac{\pi}{e} \quad (\text{L4.1})$$

**Solution:** Do the  $x = e^{\ln(x)}$  trick,

$$L = \lim_{x \rightarrow 0} \exp \left( \ln \left( \left( \frac{x^\pi - \pi^x}{x^e - e^x} \right)^{\csc(x)} \right) \right) = \exp \left( \lim_{x \rightarrow 0} \csc(x) \ln \left( \frac{x^\pi - \pi^x}{x^e - e^x} \right) \right) \quad (\text{L4.2})$$

where I dragged the limit through the exponential since it is continuous over its domain. For small  $x$ ,  $a^x$  dominates  $x^a$ , and hence I can take  $x^a - a^x \sim -a^x$  under the limit:

$$\begin{aligned} L &= \exp \left( \lim_{x \rightarrow 0} \csc(x) \ln \left( \frac{\pi^x}{e^x} \right) \right) = \exp \left( \ln \left( \frac{\pi}{e} \right) \lim_{x \rightarrow 0} x \csc(x) \right) \\ &= \exp \left( \ln \left( \frac{\pi}{e} \right) \right) \\ &= \frac{\pi}{e}, \end{aligned} \quad (\text{L4.3})$$

where I used l'Hopital's rule trivially on the last limit.