Some Tricky Problems

Ethan Huecker

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1 Integrals

1.1 Integral 1

$$I = \int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4} \tag{1}$$

Solution: Use the logarithm's properties with a subsequent I.B.P,

$$\begin{aligned} I &= \int_0^\infty \ln\left(e^x+1\right) dx - \int_0^\infty \ln\left(e^x+1\right) dx \\ &= \left[x\ln(e^x+1) - x\ln(e^x-1)\right]\Big|_0^\infty - \int_0^\infty \frac{x}{1+e^{-x}}\,dx + \int_0^\infty \frac{x}{1-e^{-x}}\,dx. \end{aligned} \tag{2}$$

Let's inspect the boundary term,

$$\begin{aligned} \lim_{x\to\infty} x\ln\left(\frac{e^x+1}{e^x-1}\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{e^x+1}{e^x-1}\right)}{1/x} = \lim_{x\to\infty}\frac{\left(\frac{e^x-1}{e^x+1}\right)\left(\frac{e^x(e^x-1)-e^x(e^x+1)}{(e^x-1)^2}\right)}{-1/x^2} &= 2\lim_{x\to\infty}\frac{x^2e^x}{(e^x+1)(e^x-1)} \\ &= 2\lim_{x\to\infty}\frac{x^2}{e^x-e^{-x}} \\ &= 0, \end{aligned} \tag{3}$$

$$\begin{aligned} \lim_{x\to 0} x\ln(e^x+1) - x\ln(e^x-1) = -\lim_{x\to 0}\frac{\ln(e^x-1)}{1/x} = -\lim_{x\to 0}\frac{x^2}{1-e^{-x}} &= -2\lim_{x\to 0}\frac{x}{e^{-x}} \\ &= 0, \end{aligned} \tag{4}$$

so the entire boundary term vanishes. Note that $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to 0} e^{-x} = 1$, so the integrands satisfy the condition to be represented as a geometric series,

$$\begin{aligned} I &= -\int_0^\infty \frac{x}{1+e^{-x}}\,dx + \int_0^\infty \frac{x}{1-e^{-x}}\,dx \\ &= \sum_{n=0}^\infty (-1)^{n+1}\int_0^\infty xe^{-nx}\,dx + \sum_{n=0}^\infty \int_0^\infty xe^{-nx}\,dx, \end{aligned} \tag{5}$$

where I commuted the integral and sum because I am a physicist. Doing a little I.B.P,

$$\begin{aligned} I &= \left(\sum_{n=0}^\infty (-1)^{n+1} + \sum_{n=0}^\infty\right)\left(\cancel{-\frac{x}{n}e^{-nx}\Big|_0^\infty} + \frac{1}{n}\int_0^\infty e^{-nx}\,dx\right) \\ &= \sum_{n=0}^\infty \frac{(-1)^{n+1}}{n^2} + \sum_{n=0}^\infty \frac{1}{n^2} \\ &= \eta(2) + \zeta(2) = \frac{\pi^2}{12} + \frac{\pi^2}{6} \implies \int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4}. \end{aligned} \tag{6}$$

1.2 Integral 2

$$\int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n} \tag{7}$$

This was derived by Johann Bernoulli in 1697.

Proof: Let's fix the nested exponent with some logarithms,

$$\begin{aligned}\int_0^1 x^{-x}\,dx &= \int_0^1 e^{\ln(x^{-x})}\,dx \\ &= \int_0^1 e^{-x\ln(x)}\,dx.\end{aligned} \tag{8}$$

Now, I need to get an infinite sum from somewhere, so how about I substitute the Maclaurin series for e^x,

$$\begin{aligned}\int_0^1 x^{-x}\,dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-x\ln(x))^n}{n!}\,dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n \ln^n(x)\,dx,\end{aligned} \tag{9}$$

where as a physicist I assumed the power series converges uniformly. This looks like it can be manipulated to a gamma function, so let $u = \ln(x)$ such that

$$\int_0^1 x^{-x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^0 e^{(n+1)u} u^n\,du. \tag{10}$$

Making the further substitution $u = -v/(n+1)$ to get the exponential how I want it,

$$\begin{aligned}\int_0^1 x^{-x}\,dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\infty}^0 \frac{(-1)^n}{(n+1)^n} v^n e^{-v} \left(-\frac{dv}{n+1}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n!(n+1)^{n+1}} \int_0^{\infty} v^n e^{-v} dv \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} \implies \int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n} \quad \square\end{aligned} \tag{11}$$

In the 2nd-to-last step I recognized the integral as $\Gamma(n+1) = n!$ and shifted the index in the last step.

1.3 Integral 3

$$I = \int_0^{1/2} \Gamma(1-x)\Gamma(1+x)\, dx = \frac{2}{\pi} G \approx 0.583 \tag{12}$$

where $G \approx 0.915$ is Catalan's constant.

Solution: The first few steps are to use the fact that $x\Gamma(x) = \Gamma(1+x)$, as well as the reflection property of the Gamma function:

$$\begin{aligned} I &= \int_0^{1/2} x\Gamma(x)\Gamma(1-x)\, dx \\ &= \int_0^{1/2} \frac{\pi x}{\sin(\pi x)}\, dx \\ &= \frac{1}{\pi}\int_0^{\pi/2} \frac{u}{\sin u}\, du. \end{aligned} \tag{13}$$

Maybe this will look better when I use the complex-exp version of sine,

$$I = \frac{2i}{\pi}\int_0^{\pi/2} \frac{u}{e^{iu} - e^{-iu}}\, du = \frac{2i}{\pi}\int_0^{\pi/2} \frac{u}{e^{iu}\left(1 - e^{-2iu}\right)}\, du. \tag{14}$$

Since $e^{-2iu} \in [-1, 1]$ on the bounds of integration $[0, \pi/2]$, it suffices to represent the term in the denominator in a geometric series,

$$I = \frac{2i}{\pi}\int_0^{\pi/2} \frac{u}{e^{iu}} \sum_{n=0}^{\infty} e^{-i2nu}\, du = \frac{2i}{\pi}\int_0^{\pi/2} \sum_{n=0}^{\infty} u e^{-iu(2n+1)}\, du \tag{15}$$

Does it makes sense to swap the integration and sum? Well, $e^{-iu(2n+1)}$ is bounded on the region of integration so I don't see why not:

$$\begin{aligned} I &= \frac{2i}{\pi}\sum_{n=0}^{\infty}\int_0^{\pi/2} u e^{-iu(2n+1)}\, du \\ &= \frac{2i}{\pi}\sum_{n=0}^{\infty}\left[-\frac{u}{i(2n+1)} e^{-iu(2n+1)}\Big|_0^{\pi/2} + \frac{1}{i(2n+1)}\int_0^{\pi/2} e^{-iu(2n+1)}\, du\right] \\ &= \frac{2i}{\pi}\sum_{n=0}^{\infty}\left[-\frac{\pi}{2i(2n+1)} e^{-i\pi(n+1/2)} + \frac{1}{(2n+1)^2} e^{-i\pi(n+1/2)} - \frac{1}{(2n+1)^2}\right] \end{aligned} \tag{16}$$

Now, it is simple to show that $e^{-i\pi(n+1/2)} = -i(-1)^n$, so

$$\begin{aligned} I &= \frac{2i}{\pi}\sum_{n=0}^{\infty}\left[\frac{\pi(-1)^n}{2(2n+1)} - \frac{i(-1)^n}{(2n+1)^2} - \frac{1}{(2n+1)^2}\right] \\ &= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} + i\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} - \frac{2i}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\ &= \frac{2}{\pi}G + i\left(\frac{\pi}{4}\right) - \frac{2i}{\pi}\left(\frac{\pi^2}{8}\right) \\ &= \frac{2}{\pi}G. \end{aligned} \tag{17}$$

$$\therefore \int_0^{1/2} \Gamma(1-x)\Gamma(1+x)\, dx = \frac{2}{\pi}G \approx 0.583 \tag{18}$$

Here, $G \approx 0.915$ is Catalan's constant. Quite nice how the two imaginary terms cancelled!

1.4 Integral 4

$$\int_{-\infty}^{\infty} \frac{e^{-x^2}\sin\left(x^2\right)}{x^2}\, dx = \sqrt{\pi(2\sqrt{2}-2)} \tag{19}$$

Solution: Looks like a standard Feynman's trick to me,

$$\begin{aligned}
\Omega(a) = \int_{-\infty}^{\infty} \frac{e^{-ax^2}\sin\left(x^2\right)}{x^2}\, dx \implies \Omega'(a) &= -\int_{-\infty}^{\infty} e^{-ax^2}\sin\left(x^2\right)\, dx \\
&= -\frac{1}{2i}\int_{-\infty}^{\infty} e^{-ax^2}\left(e^{ix^2} - e^{-ix^2}\right)\, dx \\
&= -\frac{1}{2i}\int_{-\infty}^{\infty} e^{-(a-i)x^2}\, dx + \frac{1}{2i}\int_{-\infty}^{\infty} e^{-(a+i)x^2}\, dx \\
&= \frac{\sqrt{\pi}}{2i}\left(\frac{1}{\sqrt{a+i}} - \frac{1}{\sqrt{a-i}}\right),
\end{aligned} \tag{20}$$

where I used the standard result for a Gaussian integral we all know and love. Integrating backwards for $\Omega(a)$,

$$\begin{aligned}
\Omega(1) &= \frac{\sqrt{\pi}}{2i}\lim_{a\to 1}\int\left(\frac{1}{\sqrt{a+i}} - \frac{1}{\sqrt{a-i}}\right)\, da \\
&= \frac{\sqrt{\pi}}{i}\lim_{a\to 1}\left(\sqrt{a+i} - \sqrt{a-i}\right) \\
&= \frac{\sqrt{\pi}}{i}\left(\sqrt{1+i} - \sqrt{1-i}\right) \\
&= \frac{\sqrt{\pi}}{i}\left(2^{1/4}e^{i\pi/8} - 2^{1/4}e^{-i\pi/8}\right) \\
&= 2\sqrt{\pi\sqrt{2}}\sin(\pi/8) \\
&= 2\sqrt{\pi\sqrt{2}}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right) \\
&= \sqrt{\pi(2\sqrt{2}-2)},
\end{aligned} \tag{21}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{-x^2}\sin\left(x^2\right)}{x^2}\, dx = \sqrt{\pi(2\sqrt{2}-2)} \tag{22}$$

Quite simple.

1.5 Integral 5

$$\int_{\pi/4}^{\pi/2} \ln(\ln(\tan x))\, dx = \frac{\pi}{2} \ln\left(\sqrt{2\pi} \cdot \frac{\Gamma(3/4)}{\Gamma(1/4)}\right) \tag{23}$$

This is known as the "Vardi" integral, first solved by Ilan Vardi in 1988.

Solution: The trick is to take the substitution $y = \ln(\tan x)$, from which results in a differential $\frac{dx}{dy} = e^y/(1+e^{2y})$:

$$\begin{aligned} I &= \int_0^\infty \frac{e^y}{1+e^{2y}} \ln(y)\, dy = \int_0^\infty \left(\frac{1}{1+e^{-2y}}\right) e^{-y} \ln(y)\, dy \\ &= \sum_{n=0}^{\infty} (-1)^n \int_0^\infty e^{-(2n+1)y} \ln(y)\, dy. \end{aligned} \tag{24}$$

Making the substitution $u = (2n+1)y$,

$$\begin{aligned} I = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^\infty e^{-u} \ln\left(\frac{u}{2n+1}\right)\, du &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^\infty e^{-u} \ln(u)\, du - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \ln(2n+1) \int_0^\infty e^{-u}\, du \\ &= -\gamma \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \ln(2n+1), \end{aligned} \tag{25}$$

where I recognized the integral definition of the Euler-Mascheroni constant γ. The first sum can be simplified by recalling the series expansion of inverse tangent,

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \tag{26}$$

for which we have $\tan^{-1}(1) = \pi/4$. I will also replace the second summation by an equivalent representation,

$$I = -\frac{\gamma\pi}{4} - \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \sin(n\pi/2), \tag{27}$$

where I used the fact that the terms in (25) run over odd integers and the sine factor takes those to zero anyway. To evaluate the remaining sum, I will take inspiration from the Fourier (Kummer) expansion of the log-gamma function,

$$\begin{aligned} \ln\Gamma(z) &= \left(\frac{1}{2} - z\right)(\gamma + \ln(2)) + (1-z)\ln(\pi) - \frac{1}{2}\ln(\sin(\pi z)) + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\ln(n)}{n}\sin(2\pi n z) \\ \iff -\sum_{n=1}^{\infty} \frac{\ln(n)}{n} \sin\left(\frac{n\pi}{2}\right) &= \frac{\pi}{4}(\gamma + \ln(2)) + \frac{3\pi}{4}\ln(\pi) - \frac{\pi}{2}\ln\left(\frac{\sqrt{2}}{2}\right) - \pi\ln\Gamma\left(\frac{1}{4}\right), \end{aligned} \tag{28}$$

such that (27) can be written as

$$\begin{aligned} I &= -\frac{\gamma\pi}{4} + \frac{\pi}{4}(\gamma + \ln(2)) + \frac{3\pi}{4}\ln(\pi) - \frac{\pi}{2}\ln\left(\frac{\sqrt{2}}{2}\right) - \pi\ln\Gamma\left(\frac{1}{4}\right) \\ &= \frac{\pi}{2}\left(2\ln(\sqrt{2}) + \ln(\pi^{3/2}) + \ln\left(\Gamma^{-2}\left(\frac{1}{4}\right)\right)\right) \\ &= \frac{\pi}{2}\ln\left(\frac{2\pi^{3/2}}{\Gamma^2(1/4)}\right). \end{aligned} \tag{29}$$

Using the fact that $\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$, it follows that

$$\int_{\pi/4}^{\pi/2} \ln(\ln(\tan x))\, dx = \frac{\pi}{2} \ln\left(\sqrt{2\pi} \cdot \frac{\Gamma(3/4)}{\Gamma(1/4)}\right) \tag{30}$$

is the solution to the Vardi integral.

1.6 Integral 6

$$I=\int_0^{\pi/2}\frac{\{\tan(x)\}}{\tan(x)}\,dx=\frac{1}{2}\left(\pi-\ln\left(\frac{\sinh(\pi)}{\pi}\right)\right) \tag{31}$$

Here, $\{\tan(x)\}$ is the fractional part of $\tan(x)$.

Solution: The fractional part satisfies $\{\tan(x)\}=\tan(x)-\lfloor\tan(x)\rfloor$, so

$$I=\int_0^{\pi/2}1-\frac{\lfloor\tan(x)\rfloor}{\tan(x)}\,dx=\frac{\pi}{2}-\int_0^{\pi/2}\frac{\lfloor\tan(x)\rfloor}{\tan(x)}\,dx. \tag{32}$$

Letting $u=\tan(x)$,

$$\begin{aligned}I=\frac{\pi}{2}-\int_0^{\infty}\frac{\lfloor u\rfloor}{u(1+u^2)}\,du&=\frac{\pi}{2}-\lim_{n\to\infty}\left[\int_0^1\frac{0}{u(1+u^2)}\,du+\int_1^2\frac{1}{u(1+u^2)}\,du+\cdots+\int_{n-1}^{n}\frac{n-1}{u(1+u^2)}\,du\right]\\&=\frac{\pi}{2}-\sum_{j=1}^{\infty}j\int_j^{j+1}\frac{du}{u(1+u^2)}\\&=\frac{\pi}{2}-\sum_{j=1}^{\infty}j\int_{\tan^{-1}(j)}^{\tan^{-1}(j+1)}\cot(x)\,dx.\end{aligned} \tag{33}$$

Since $\frac{d}{dx}\ln(\sin(x))=\cot(x)$,

$$\begin{aligned}I&=\frac{\pi}{2}-\sum_{j=1}^{\infty}j\left(\ln\left(\sin\tan^{-1}(j+1)\right)-\ln\left(\sin\tan^{-1}(j)\right)\right)\\&=\frac{\pi}{2}-\sum_{j=1}^{\infty}\left((j+1)\ln\left(\frac{j+1}{\sqrt{1+(j+1)^2}}\right)-j\ln\left(\frac{j}{\sqrt{1+j^2}}\right)\right)+\sum_{j=1}^{\infty}\ln\left(\frac{j+1}{\sqrt{1+(j+1)^2}}\right)\end{aligned} \tag{34}$$

Where I added a zero in order for the series to telescope properly. It is standard to show that the middle series has the partial sum

$$S_n=-\ln\left(\frac{1}{\sqrt{2}}\right)+(n+1)\ln\left(\frac{n+1}{\sqrt{1+(n+1)^2}}\right)\implies\lim_{n\to\infty}S_n=\frac{\ln(2)}{2}, \tag{35}$$

such that

$$I=\frac{\pi}{2}-\frac{\ln(2)}{2}+\sum_{j=1}^{\infty}\ln\left(\frac{j+1}{\sqrt{1+(j+1)^2}}\right)=\frac{\pi}{2}-\frac{\ln(2)}{2}-\frac{1}{2}\sum_{j=1}^{\infty}\ln\left(1+\frac{1}{(j+1)^2}\right), \tag{36}$$

where I did some manipulations in the second equality. The remaining series does not telescope, and I will rather look at the Euler product formula for sine,

$$\frac{\sin(x)}{x}=\prod_{j=1}^{\infty}\left(1-\frac{x^2}{\pi^2j^2}\right)\implies\ln\left(\frac{\sin(x)}{x}\right)=\sum_{j=1}^{\infty}\ln\left(1-\frac{x^2}{\pi^2j^2}\right). \tag{37}$$

Substituting $x=i\pi$,

$$\ln\left(\frac{\sin(i\pi)}{i\pi}\right)=\sum_{j=1}^{\infty}\ln\left(1+\frac{1}{j^2}\right)=\ln(2)+\sum_{j=1}^{\infty}\ln\left(1+\frac{1}{(j+1)^2}\right) \tag{38}$$

Solving this equation for the series and substituting it into (36) results in

$$I=\frac{\pi}{2}-\frac{\ln(2)}{2}+\frac{\ln(2)}{2}-\frac{1}{2}\ln\left(\frac{\sin(i\pi)}{i\pi}\right)=\frac{1}{2}\left(\pi-\ln\left(\frac{\sinh(\pi)}{\pi}\right)\right). \tag{39}$$

1.7 Integral 7

$$I = \int_0^1 \int_0^1 \frac{\ln^s(x)}{1-xy}\, dxdy = (-1)^s \Gamma(s+1)\zeta(s+2) \tag{40}$$

Solution: The integral over y is really easy:

$$\begin{aligned} I &= \int_0^1 \ln^s(x) \left[\int_0^1 \frac{dy}{1-xy}\right] dx \\ &= -\int_0^1 \ln^s(x) \Big[\ln(1-xy)\Big]_0^1 \frac{dx}{x} \\ &= -\int_0^1 \ln^s(x) \ln(1-x)\, \frac{dx}{x}. \end{aligned} \tag{41}$$

The trick I'll employ is to use the series expansion for $\ln(1-x)$,

$$\begin{aligned} I &= -\int_0^1 \ln^s(x) \left(-\sum_{n=1}^{\infty} \frac{x^n}{n}\right) \frac{dx}{x} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n \ln^s(x) \frac{dx}{x}. \end{aligned} \tag{42}$$

Substituting $u = -n\ln(x)$,[1]

$$\begin{aligned} I &= (-1)^s \sum_{n=1}^{\infty} \frac{1}{n^{s+2}} \int_0^{\infty} u^s e^{-u} du \\ &= (-1)^s \Gamma(s+1) \sum_{n=1}^{\infty} \frac{1}{n^{s+2}} \\ &= (-1)^s \Gamma(s+1)\zeta(s+2), \end{aligned} \tag{43}$$

$$\therefore \int_0^1 \int_0^1 \frac{\ln^s(x)}{1-xy}\, dxdy = (-1)^s \Gamma(s+1)\zeta(s+2) \tag{44}$$

which is quite pretty.

[1] I did like three substitutions in a row on paper and I'm doing them all at once now that I have the hindsight.

1.8 Integral 8

$$I = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \zeta(s)\Gamma(s) \tag{45}$$

Solution: Rewrite so that I can expand in a series:

$$\begin{aligned} I = \int_0^\infty \frac{x^{s-1}}{e^x-1}\, dx &= \int_0^\infty \frac{x^{s-1}e^{-x}}{1-e^{-x}}\, dx \\ &= \int_0^\infty x^{s-1}e^{-x}\sum_{n=0}^\infty e^{-nx}\, dx \\ &= \sum_{n=0}^\infty \int_0^\infty x^{s-1}e^{-(n+1)x}\, dx. \end{aligned} \tag{46}$$

Doing the substitution $u = (n+1)x$,

$$\begin{aligned} I = \sum_{n=0}^\infty \frac{1}{(n+1)^s}\int_0^\infty u^{s-1}e^{-u}\, du &= \Gamma(s)\sum_{n=1}^\infty \frac{1}{n^s} \\ &= \zeta(s)\Gamma(s) \implies \int_0^\infty \frac{x^{s-1}}{e^x-1}\, dx = \zeta(s)\Gamma(s) \end{aligned} \tag{47}$$

Pretty easy. Thie integrals shown up often in statistical mechanics, since the exponential term is reminiscent of the Bose-distribution.

1.9 Integral 9

$$I = \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1}\, dx = \Gamma(n)\mathrm{Li}_n(z) \tag{48}$$

Solution: Let's rewrite an expand the integrand in a power series,

$$\begin{aligned} I = z\int_0^\infty \frac{x^{n-1}e^{-x}}{1-ze^{-x}}\, dx &= z\int_0^\infty x^{n-1}e^{-x}\sum_{m=0}^\infty z^m e^{-mx}\, dx \\ &= \sum_{m=0}^\infty z^{m+1}\int_0^\infty x^{n-1}e^{-(m+1)x}\, dx \\ &= \sum_{m=0}^\infty \frac{z^{m+1}}{(m+1)^n}\int_0^\infty u^{n-1}e^{-u}\, du \\ &= \Gamma(n)\sum_{m=1}^\infty \frac{z^m}{m^n} = \Gamma(n)\mathrm{Li}_n(z) \implies \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x-1}\, dx = \Gamma(n)\mathrm{Li}_n(z). \end{aligned} \tag{49}$$

1.10 Integral 10

$$I = \int_0^{2\pi} d\phi\, \frac{1}{s^2 - 2sR\cos\phi + R^2} = \frac{2\pi}{|s^2 - R^2|} \tag{50}$$

This shows up commonly in many electrodynamics problems. Note that the denominator is the expansion of $||s\hat{r} - R\hat{r}'||^2$ in cylindrical coordinates in the case that $z = z' = \phi' = 0$.

Solution: Define the integral

$$I' = \int_0^{2\pi} d\phi\, \frac{1}{a + b\cos\phi}. \tag{51}$$

Clearly $|a| > |b|$ for finite-ness. Make the substitution that $z = e^{i\phi}$, such that $dz = iz\; d\phi$ and $2\cos\phi = z + z^{-1}$. This transforms the integration over $\phi \in \{0, 2\pi\}$ to an integration over the unit circle C in the complex plane:

$$\begin{aligned} I' = \oint \frac{dz}{iz} \frac{1}{a + \frac{b}{2}(z + z^{-1})} &= \frac{2}{ib} \oint dz\, \frac{1}{z^2 + \frac{2a}{b}z + 1} \\ &= \frac{2}{ib} \oint dz\, \frac{1}{(z - z_+)(z - z_-)}, \end{aligned} \tag{52}$$

where z_+, z_- are the poles of the integrand:

$$z_\pm = \frac{a}{b}\left[-1 \pm \sqrt{1 - \frac{b^2}{a^2}}\right] \tag{53}$$

Only z_+ is inside of the unit circle, and since it is a simple pole, Cauchy's Theorem tells me that the line integral (52) is $2\pi i$ times that residue,

$$\begin{aligned} I' = \frac{2}{ib}(2\pi i) \lim_{z \to z_+} \frac{(z - z_+)}{(z - z_+)(z - z_-)} &= \frac{4\pi}{b} \frac{1}{z_+ - z_-} \\ &= \frac{4\pi}{b} \frac{1}{\frac{2a}{b}\sqrt{1 - \frac{b^2}{a^2}}} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}}. \end{aligned} \tag{54}$$

This is already a neat result in itself. In the case that $a = s^2 + R^2$ and $b = -2sR$, I find

$$\int_0^{2\pi} d\phi\, \frac{1}{s^2 - 2sR\cos\phi + R^2} = \frac{2\pi}{\sqrt{s^4 - 2s^2R^2 + R^4}} = \frac{2\pi}{|s^2 - R^2|} \tag{55}$$

1.11 Integral 11

Derive the following contour integral representation of the binomial coefficient,

$$I = \frac{1}{2\pi i} \oint_C dz \, \frac{(1+z)^y}{z^{x+1}} = \binom{y}{x}, \tag{56}$$

where C is the circle $|z| = 1$ traced CCW.

Solution: First let $z \to -z$,

$$I = \frac{1}{2\pi i} \frac{1}{(-1)^x} \oint_C dz \, \frac{(1-z)^y}{z^{x+1}} = \frac{e^{i\pi x}}{2\pi i} \oint_C dz \, \frac{(1-z)^y}{z^{x+1}}, \tag{57}$$

then parameterize C via $z = e^{i\theta}$, $d\theta = dz/iz$ such that

$$I = \frac{e^{i\pi x}}{2\pi} \int_0^{2\pi} d\theta \, \frac{\left(1 - e^{i\theta}\right)^y}{e^{ix\theta}}. \tag{58}$$

The now trick is to use the binomial theorem,

$$(a+b)^c = \sum_{j=0}^{c} \binom{c}{j} a^{c-j} b^j \implies \left(1 - e^{i\theta}\right)^y = \sum_{j=0}^{y} \binom{y}{j} (-1)^j e^{i\theta j}, \tag{59}$$

such that

$$\begin{aligned} I = \frac{e^{i\pi x}}{2\pi} \sum_{j=0}^{y} \binom{y}{j} (-1)^j \int_0^{2\pi} d\theta \, e^{i(j-x)\theta} &= \frac{e^{i\pi x}}{2\pi i} \sum_{j=0}^{y} \binom{y}{j} (-1)^j \frac{1}{j-x} \left(e^{2\pi i(j-x)} - 1\right) \\ &= \frac{e^{-i\pi x} - e^{i\pi x}}{2\pi i} \sum_{j=0}^{y} \binom{y}{j} (-1)^j \frac{1}{j-x} \\ &= -\frac{\sin(\pi x)}{\pi} \sum_{j=0}^{y} \binom{y}{j} (-1)^j \frac{1}{j-x}, \end{aligned} \tag{60}$$

where I noted $e^{2\pi i(j-x)} = e^{-2\pi i x}$ since $j \in \mathbb{Z}$. Using the representation $(j-x)^{-1} = \int_0^1 dt \, t^{j-x-1}$,

$$\begin{aligned} I = -\frac{\sin(\pi x)}{\pi} \sum_{j=0}^{y} \binom{y}{j} (-1)^j \int_0^1 dt \, t^{j-x-1} &= -\frac{\sin(\pi x)}{\pi} \int_0^1 dt \, \left[\sum_{j=0}^{y} \binom{y}{j} (-t)^j\right] t^{-x-1} \\ &= -\frac{\sin(\pi x)}{\pi} \int_0^1 dt \, (1-t)^y t^{-x-1} \\ &= -\frac{\sin(\pi x)}{\pi} B(-x, 1+y), \end{aligned} \tag{61}$$

where in the third equality I used the binomial theorem, after which I recognized a variant on the beta function. By the gamma function reflection formula and the properties of the beta function,

$$-\frac{\sin(\pi x)}{\pi} = \frac{1}{\Gamma(-x)\Gamma(1+x)}, \qquad B(-x, 1+y) = \frac{\Gamma(-x)\Gamma(1+y)}{\Gamma(1+y-x)}, \tag{62}$$

I find that

$$\begin{aligned} I = \frac{1}{\Gamma(-x)\Gamma(1+x)} \frac{\Gamma(-x)\Gamma(1+y)}{\Gamma(1+y-x)} &= \frac{\Gamma(1+y)}{\Gamma(1+x)\Gamma(1+y-x)} \\ &= \frac{y!}{x!(y-x)!} \implies \frac{1}{2\pi i} \oint_C dz \, \frac{(1+z)^y}{z^{x+1}} = \binom{y}{x} \end{aligned} \tag{63}$$

as desired.

1.12 Integral 12

$$I = \int_0^1 dx \left\{\frac{1}{x}\right\} = 1 - \gamma \tag{64}$$

Here $\{x\}$ is the fractional part of x.

Solution: Let $y = 1/x$ such that $dx = -dy/y^2$ and

$$\begin{aligned} I = \int_1^\infty dy \, \frac{\{y\}}{y^2} &= \int_0^\infty dy \, \frac{dy}{y} - \int_1^\infty dy \, \frac{\lfloor y \rfloor}{y^2} \\ &= \lim_{N\to\infty} \ln(N) - \left[\int_1^2 dy \, \frac{1}{y^2} + \int_2^3 dy \, \frac{2}{y^2} + \cdots\right] \\ &= \lim_{N\to\infty} \ln(N) - \lim_{N\to\infty} \sum_{n=1}^N \int_n^{n+1} dy \, \frac{n}{y^2} \\ &= \lim_{N\to\infty} \ln(N) - \lim_{N\to\infty} \sum_{n=1}^N \frac{1}{n+1} \\ &= 1 + \lim_{N\to\infty} \left[\ln(N) - \lim_{N\to\infty} \sum_{n=1}^N \frac{1}{n}\right] = 1 - \gamma \implies \int_0^1 dx \left\{\frac{1}{x}\right\} = 1 - \gamma \end{aligned} \tag{65}$$

where I noted $\{x\} = x - \lfloor x \rfloor$, and recognized the Euler-Mascheroni constant in its form as the limiting difference between the harmonic series and the natural logarithm.

1.13 Integral 13

Evaluate the following integral,

$$I = \int_{-\pi/2}^{\pi/2} dx \, \frac{\cos(x)}{e^x + 1}, \tag{66}$$

by first proving the integral identity

$$\int_{-a}^{a} dx \, \frac{E(x)}{k(x)+1} = \frac{1}{2} \int_{-a}^{a} dx \, E(x), \tag{67}$$

where $E(x)$ is even for $x \in [-a, a]$ and $k(x)k(-x) = 1$.[2]

Solution: Let $x \to -x$ in (67):

$$\int_{-a}^{a} dx \, \frac{E(x)}{k(x)+1} = \int_{-a}^{a} dx \, \frac{E(x)}{k(-x)+1}. \tag{68}$$

Since these are equivalent, it follows that

$$\begin{aligned} \int_{-a}^{a} dx \, \frac{E(x)}{k(x)+1} = \frac{1}{2} \int_{-a}^{a} dx \left[\frac{E(x)}{k(x)+1} + \frac{E(x)}{k(-x)+1}\right] &= \frac{1}{2} \int_{-a}^{a} dx \, E(x) \left[\frac{k(x)+k(-x)+2}{k(x)k(-x)+k(x)+k(-x)+1}\right] \\ &= \frac{1}{2} \int_{-a}^{a} dx \, E(x) \quad \square \end{aligned} \tag{69}$$

and the identity is proved. For (66), I find

$$\int_{-\pi/2}^{\pi/2} dx \, \frac{\cos(x)}{e^x+1} = \frac{1}{2} \int_{-\pi/2}^{\pi/2} dx \, \cos(x) = 1. \tag{70}$$

[2] I found this integral identity on instagram. The author didn't provide the proof, so I proved it and provided an additional example.

1.14 Integral 14

$$\int_0^\infty dx\, \frac{e^{-ax} - e^{-bx}}{x\sec(px)} = \frac{1}{2}\ln\left(\frac{b^2+p^2}{a^2+p^2}\right) \tag{71}$$

Here $\mathrm{Re}(a), \mathrm{Re}(b) > |\mathrm{Im}(p)|$.

Solution: Let's go.

$$\begin{aligned}
I &= \int_0^\infty dx\, \frac{1}{x}\cos(px)e^{-ax} - \int_0^\infty dx\, \frac{1}{x}\cos(px)e^{-bx} \\
&= \frac{1}{2}\int_0^\infty dx\, \frac{1}{x}\left(e^{(ip-a)x} + e^{-(ip+a)x}\right) - \frac{1}{2}\int_0^\infty dx\, \frac{1}{x}\left(e^{(ip-b)x} + e^{-(ip+b)x}\right) \\
&= -\frac{1}{2}\int_0^a dy \int_0^\infty dx\, \left(e^{(ip-y)x} + e^{-(ip+y)x}\right) + \frac{1}{2}\int_0^b dy \int_0^\infty dx\, \left(e^{(ip-y)x} + e^{-(ip+y)x}\right) \\
&= -\frac{1}{2}\int_0^a dy \left(\frac{1}{y-ip} + \frac{1}{y+ip}\right) + \frac{1}{2}\int_0^b dy \left(\frac{1}{y-ip} + \frac{1}{y+ip}\right) \\
&= \frac{1}{2}\int_a^b dy\, \frac{2y}{y^2+p^2} \\
&= \frac{1}{2}\int_{a^2+p^2}^{b^2+p^2} du\, \frac{1}{u} \\
&= \frac{1}{2}\ln\left(\frac{b^2+p^2}{a^2+p^2}\right). \quad \square
\end{aligned} \tag{72}$$

1.15 Integral 15

$$\iint_{[0,1]^2} dxdy\sqrt{\frac{x}{y}+\frac{y}{x}} = \frac{2\sqrt{2}}{3} + \frac{\Gamma^2(1/4)}{6\sqrt{\pi}} \tag{73}$$

Solution: Substitute $t = x/y$, $dx = ydt$, then perform an IBP over y using the FTC-1:

$$\begin{aligned} I = \int_0^1 dy\; y \int_0^{1/y} dt\; \sqrt{t+\frac{1}{t}} &= \frac{y^2}{2}\int_0^{1/y} dt\; \sqrt{t+\frac{1}{t}}\Big|_0^1 + \frac{1}{2}\int_0^1 dy\; \sqrt{y+\frac{1}{y}} \\ &= \frac{1}{2}\int_0^1 dt\; \sqrt{t+\frac{1}{t}} + \frac{1}{2}\int_0^1 dy\; \sqrt{y+\frac{1}{y}} \\ &= \int_0^1 dt\; \sqrt{t+\frac{1}{t}}, \end{aligned} \tag{74}$$

where the lower bound $y \to 0$ vanishes by noting that the integral will behave like $y^{-3/2}$, and so $y^2 y^{-3/2} = y^{1/2} \to 0$. The trick here is to make the substitution $y = u^2$ and perform another IBP,

$$\begin{aligned} I = 2\int_0^1 dt\; \sqrt{1+u^4} &= 2u\sqrt{1+u^4}\Big|_0^1 - 4\int_0^1 du\; \frac{u^4}{\sqrt{1+u^4}} \\ &= 2\sqrt{2} - 4\int_0^1 du\; \sqrt{1+u^4} + 4\int_0^1 du\; \frac{1}{\sqrt{1+u^4}}. \end{aligned} \tag{75}$$

I find the same integral on the r.h.s as the l.h.s, and hence

$$I = \frac{2\sqrt{2}}{3} + \frac{4}{3}\int_0^1 du\; \frac{1}{\sqrt{1+u^4}}. \tag{76}$$

Inspired by the beta function, substitute $u^2 = \tan(v)$,

$$\begin{aligned} I = \frac{2\sqrt{2}}{3} + \frac{2}{3}\int_0^{\pi/4} dv\; \frac{1}{\sqrt{\sin(v)\cos(v)}} &= \frac{2\sqrt{2}}{3} + \frac{2\sqrt{2}}{3}\int_0^{\pi/4} dv\; \frac{1}{\sqrt{\sin(2v)}} \\ &= \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{3}\int_0^{\pi/2} dw\; \sin^{-1/2}(w)\cos^0(w) \\ &= \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{6} B\left(1/4, 1/2\right) \\ &= \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{6}\frac{\Gamma^2(1/4)\Gamma(1/2)}{\Gamma(1/4)\Gamma(1/4+1/2)}. \end{aligned} \tag{77}$$

Using the Legendre duplication formula,

$$\Gamma(1/4)\Gamma(1/4+1/2) = 2^{1-2(1/4)}\sqrt{\pi}\Gamma(2(1/4)) = \sqrt{2\pi}\Gamma(1/2), \tag{78}$$

and so

$$\iint_{[0,1]^2} dxdy\sqrt{\frac{x}{y}+\frac{y}{x}} = \frac{2\sqrt{2}}{3} + \frac{\Gamma^2(1/4)}{6\sqrt{\pi}} \tag{79}$$

as desired.

1.16 Integral 16

$$\int_0^\infty dx \int_0^\infty dy\, e^{-(x^n+y^n)} \ln\left[\frac{(xy)^{(xy)^n}}{(x^{x^n}y^{y^n})^{(xy)^n}}\right] = -\frac{2}{n^6}\Gamma^2\left(\frac{1}{n}\right)\left[2n+\psi\left(\frac{1}{n}\right)\right] \tag{80}$$

Solution: The log can be simplified as

$$\ln\left[\frac{(xy)^{(xy)^n}}{(x^{x^n}y^{y^n})^{(xy)^n}}\right] = x^n y^n \ln x + x^n y^n \ln y - x^{2n} y^n \ln x - x^n y^{2n} \ln y, \tag{81}$$

and so

$$\begin{aligned} I &= \int_0^\infty dx \int_0^\infty dy\, e^{-(x^n+y^n)} \left[x^n y^n \ln x + x^n y^n \ln y - x^{2n} y^n \ln x - x^n y^{2n} \ln y\right] \\ &= 2\int_0^\infty dx \int_0^\infty dy\, e^{-(x^n+y^n)} \left[x^n y^n \ln x - x^{2n} y^n \ln x\right], \end{aligned} \tag{82}$$

where I used the fact that the integrand is symmetric under $x \leftrightarrow y$. The y-integral is simple after the substitution $u = y^n$, in which

$$\begin{aligned} I &= \frac{2}{n}\int_0^\infty dx\, e^{-x^n}\left[x^n \ln x \int_0^\infty du\, u^{1/n} e^{-u} - x^{2n} \ln x \int_0^\infty du\, u^{1/n} e^{-u}\right] \\ &= \frac{2}{n}\Gamma\left(\frac{1}{n}+1\right)\int_0^\infty dx \left[\ln x\, x^n e^{-x^n} - \ln x\, x^{2n} e^{-x^n}\right]. \end{aligned} \tag{83}$$

Note that $\Gamma'(z) = \psi(z)\Gamma(z)$, where

$$\Gamma'(z) = \frac{d}{dz}\int_0^\infty dx\, x^{z-1}e^{-x} = \int_0^\infty dx\, \ln x\, x^{z-1} e^{-x}, \tag{84}$$

which motivates in (83) the substitution $u = x^n$, giving

$$\begin{aligned} I &= \frac{2}{n^3}\Gamma\left(\frac{1}{n}\right)\int_0^\infty du \left[\ln(u^{1/n})\, u^{1/n} e^{-u} - \ln(u^{1/n})\, u^{1/n+1} e^{-u}\right] \\ &= \frac{2}{n^4}\Gamma\left(\frac{1}{n}\right)\left[\Gamma'\left(\frac{1}{n}+1\right) - \Gamma'\left(\frac{1}{n}+2\right)\right] \\ &= \frac{2}{n^4}\Gamma\left(\frac{1}{n}\right)\left[\psi\left(\frac{1}{n}+1\right)\Gamma\left(\frac{1}{n}+1\right) - \psi\left(\frac{1}{n}+2\right)\Gamma\left(\frac{1}{n}+2\right)\right] \\ &= \frac{2}{n^5}\Gamma^2\left(\frac{1}{n}\right)\left[\psi\left(\frac{1}{n}+1\right) - \left(\frac{1}{n}+1\right)\psi\left(\frac{1}{n}+2\right)\right]. \end{aligned} \tag{85}$$

Using the fact that $\psi(z+1) = \psi(z) + \frac{1}{z}$, then

$$\begin{aligned} I &= \frac{2}{n^5}\Gamma^2\left(\frac{1}{n}\right)\left[\psi\left(\frac{1}{n}\right) + n - \left(\frac{1}{n}+1\right)\left(\psi\left(\frac{1}{n}\right) + n + \frac{1}{\frac{1}{n}+1}\right)\right] \\ &= \frac{2}{n^5}\Gamma^2\left(\frac{1}{n}\right)\left[-\frac{1}{n}\psi\left(\frac{1}{n}\right) - 2\right] \\ &= -\frac{2}{n^6}\Gamma^2\left(\frac{1}{n}\right)\left[2n+\psi\left(\frac{1}{n}\right)\right], \quad \square \end{aligned} \tag{86}$$

as desired.

1.17 Integral 17

$$\int_0^\infty \frac{dx}{x^n+1} = \frac{\pi}{n\sin(\pi/n)} \tag{87}$$

Solution: Let $u = x^n$, such that $dx = \frac{1}{n}u^{1/n-1}du$ and

$$I_n = \frac{1}{n}\int_0^\infty du\ \frac{u^{1/n-1}}{1+u}. \tag{88}$$

Recall the Beta function

$$B(z_1,z_2) = \int_0^\infty dt\ \frac{t^{z_1-1}}{(1+t)^{z_1+z_2}} = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}. \tag{89}$$

Comparing, I find that $z_1 = 1/n$ and $z_1 + z_2 = 1$. Hence,

$$I_n = \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{n}-1\right)}{n\Gamma(1)} = \frac{\pi}{n\sin(\pi/n)} \tag{90}$$

where I made use of Euler's reflection formula

1.18 Integral 18

$$I = \int_0^\infty dx\ \tan^{-1}(x^{-k}) = \frac{\pi}{2}\sec\left(\frac{\pi}{2k}\right) \tag{91}$$

Here $k > 1$. I found this guy on Youtube (Maths505), but the solution is my own.

Solution: Like any arctangent integral, IBP gives

$$I = x\tan^{-1}(x^{-k})\Big|_0^\infty + k\int_0^\infty \frac{dx\ x}{x^{k+1}(1+x^{-2k})} = k\int_0^\infty \frac{dx}{x^k(1+x^{-2k})}. \tag{92}$$

Letting $u = x^{1-k}$ results in

$$I = \frac{k}{k-1}\int_0^\infty \frac{du}{1+u^{2k/(k-1)}}, \tag{93}$$

where my goal now is to express this in terms of the Beta function via a second substitution $v = u^{2k/(k-1)}$, giving

$$\begin{aligned} I = \frac{1}{2}\int_0^\infty dv\ \frac{v^{-(k+1)/2k}}{1+v} &= \frac{1}{2}B\left(\frac{k-1}{2k}, 1-\frac{k-1}{2k}\right) \\ &= \frac{\pi}{2\sin(\pi(k+1)/2k)} \implies I = \frac{\pi}{2}\sec\left(\frac{\pi}{2k}\right), \quad k>1 \end{aligned} \tag{94}$$

where I used Euler's reflection formula. The Maths505 channel gives an interesting case for $k = 5$, where since $\cos(\pi/5) = \phi/2$, where ϕ is the Golden ratio, we have

$$\int_0^\infty dx\ \tan^{-1}\left(\frac{1}{x^5}\right) = \frac{\pi}{\phi}, \tag{95}$$

which is pretty cool.

1.19 Integral 19

$$I = \int_0^\infty dx\, \frac{1}{x}\left(\frac{1}{e^x} + \frac{\sin x}{1-e^x}\right) = \mathrm{Arg}(i!) \tag{96}$$

Solution: Whenever I see something like $i!$ I immediately think of the Gamma function. Here,

$$i! = \int_0^\infty dx\, x^i e^{-x}. \tag{97}$$

I will just take the principal branch, so

$$\mathrm{Arg}(i!) = \mathrm{Im}\ln(i!) = \mathrm{Im}\ln\Gamma(i+1) = \frac{\ln\Gamma(i+1) - \ln\Gamma(-i+1)}{2i}. \tag{98}$$

Using Wikipedia, the integral representation of $\ln\Gamma$ is

$$\ln\Gamma(z+1) = \int_0^\infty dt\, \frac{e^{-zt} - ze^{-t} - 1 + z}{t(e^t-1)}, \tag{99}$$

such that

$$\begin{aligned} \ln\Gamma(i+1) - \ln\Gamma(-i+1) &= \int_0^\infty dt\, \frac{e^{-it} - ie^{-t} - 1 + i - e^{it} - ie^{-t} + 1 + i}{t(e^t-1)} \\ &= \int_0^\infty dt\, \frac{-2i\sin t - 2ie^{-t} + 2i}{t(e^t-1)}, \end{aligned} \tag{100}$$

leading to the answer

$$\mathrm{Arg}(i!) = \int_0^\infty dt\, \frac{1 - e^{-t} - \sin t}{t(e^t-1)} = \int_0^\infty dt\, \frac{1}{t}\left[\frac{1}{e^t} - \frac{\sin t}{e^t-1}\right] \quad \square$$

1.20 Integral 20

$$I = \int_0^{4\pi} dx \sum_{n=1}^{\lceil \frac{x}{\pi} \rceil} (i\sin x)^n = -\frac{9\pi}{8}. \tag{101}$$

Solution: Expand the integration into four integrals with boundaries which are multiples of π,

$$\begin{aligned} I &= \int_0^{\pi} dx \sum_{n=1}^{\lceil \frac{x}{\pi} \rceil} (i\sin x)^n + \int_\pi^{2\pi} dx \sum_{n=1}^{\lceil \frac{x}{\pi} \rceil} (i\sin x)^n + \int_{2\pi}^{3\pi} dx \sum_{n=1}^{\lceil \frac{x}{\pi} \rceil} (i\sin x)^n + \int_{3\pi}^{4\pi} dx \sum_{n=1}^{\lceil \frac{x}{\pi} \rceil} (i\sin x)^n \\ &= \int_0^{\pi} dx \sum_{n=1}^{1} (i\sin x)^n + \int_\pi^{2\pi} dx \sum_{n=1}^{2} (i\sin x)^n + \int_{2\pi}^{3\pi} dx \sum_{n=1}^{3} (i\sin x)^n + \int_{3\pi}^{4\pi} dx \sum_{n=1}^{4} (i\sin x)^n. \end{aligned} \tag{102}$$

Now, let's combine terms with common powers of sine,

$$\begin{aligned} I &= \int_0^{4\pi} dx\, i\sin(x) - \int_\pi^{4\pi} dx\, \sin^2(x) - i\int_{2\pi}^{4\pi} dx\, \sin^3(x) + \int_{3\pi}^{4\pi} dx\, \sin^4(x) \\ &= -\int_\pi^{4\pi} dx\, \sin^2(x) + \int_{3\pi}^{4\pi} dx\, \sin^4(x) \\ &= -\frac{3\pi}{2} + \frac{3\pi}{8} \\ &= -\frac{9\pi}{8} \quad \square \end{aligned}$$

2 Ordinary Differential Equations

2.1 ODE 1

$$\frac{dy}{dx}\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 = 2\left(\frac{d^2y}{dx^2}\right)^2 \implies y(x) = c_1 + c_2 \tanh^{-1}(c_3 e^x) \tag{103}$$

Solution: The lowest order derivative is 1st-order, so substitute $\frac{dy}{dx} = u$, such that

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = u\frac{du}{dy}, \qquad \frac{d^3y}{dx^3} = \frac{du}{dx}\frac{du}{dy} + u\frac{d^2u}{dxdy} = u\left(\frac{du}{dy}\right)^2 + u^2\frac{d^2u}{dy^2}. \tag{104}$$

are the transformations of the higher order derivatives. Thus, (1.1) becomes

$$\begin{aligned} u^2\left[\left(\frac{du}{dy}\right)^2 + u\frac{d^2u}{dy^2}\right] + u^2 &= 2u^2\left(\frac{du}{dy}\right)^2 \\ u\frac{d^2u}{dy^2} &= \left(\frac{du}{dy}\right)^2 - 1, \end{aligned} \tag{105}$$

and we have successfully dropped an order. Continuing with this process, let $\frac{du}{dy} = v$ such that $\frac{d^2u}{dy^2} = v\frac{dv}{du}$ and (1.3) becomes

$$\begin{aligned} uv\frac{dv}{du} = v^2 - 1 \iff \int \frac{v\ dv}{v^2-1} = \int \frac{du}{u} \iff \frac{1}{2}\ln|v^2-1| &= \ln|u| + C \\ \ln\left|\frac{v^2-1}{u^2}\right| = C \implies \frac{v^2-1}{u^2} &= C, \end{aligned} \tag{106}$$

where I changed the arbitrary constant $C \in \mathbb{R}$ many times. Solving this equation for $v = \frac{du}{dy}$ gives

$$\begin{aligned} \frac{du}{dy} = \sqrt{1+Cu^2} \iff \int \frac{du}{\sqrt{1+Cu^2}} = \int dy \iff \frac{1}{\sqrt{C}}\sinh^{-1}(\sqrt{C}u) &= y + D \\ u &= \frac{1}{C}\sinh(Cy+D), \end{aligned} \tag{107}$$

where I changed the constant $\sqrt{C} \to C$ and absorbed it into the second constant $D \in \mathbb{R}$. Since we know that $u = \frac{dy}{dx}$, we actually have

$$\begin{aligned} \frac{dy}{dx} = \frac{1}{C}\sinh(Cy+D) \iff \int \operatorname{csch}(Cy+D)\ dy = \frac{1}{C}\int dx \ \to \ \frac{1}{C}\int \operatorname{csch} z\ dz &= \frac{1}{C}x + E \\ \int \frac{dz}{e^z - e^{-z}} &= \frac{x}{2} + E. \end{aligned} \tag{108}$$

Where I absorbed arbitrary constants into one another and defined $z = Cy + D$ to simplify the remaining integral. Continuing, multiply the top and bottom of the integrand by e^z to make use of a u-sub $\zeta = e^z$,

$$\begin{aligned} \frac{x}{2} + E = \int \frac{e^z\ dz}{e^{2z}-1} \quad \to \quad \int \frac{d\zeta}{\zeta^2-1} &= \frac{1}{2}\int \frac{d\zeta}{\zeta-1} - \frac{1}{2}\int\frac{d\zeta}{\zeta+1} \\ &= \frac{1}{2}\ln|\zeta-1| - \frac{1}{2}\ln|\zeta+1|\,. \end{aligned} \tag{109}$$

Making use of a trivial log property and multiplying by 2 (absorbing it into constants),

$$x + E = \ln\left|\frac{\zeta-1}{\zeta+1}\right| = \ln\left|\frac{e^z-1}{e^z+1}\right| = \ln|\tanh(z)| = \ln|\tanh(Cy+D)|. \tag{110}$$

Taking the exponential and the hyperbolic arc-tangent gives

$$\tanh(Cy+D) = Ee^x \implies Cy + D = \tanh(Ee^x) \implies y(x) = c_1 + c_2\tanh^{-1}(c_3 e^x) \tag{111}$$

which is the solution for arbitrary constants c_1, c_2, c_3.

2.2 ODE 2

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 4y^2\ln(y) \implies y(x) = e^{c_1 e^{2x} + c_2 e^{-2x}} \tag{112}$$

Solution: Make the substitution $u = \frac{dy}{dx}$, from which

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{dy}{dx}\frac{du}{dy} = u\frac{du}{dy}, \tag{113}$$

such that

$$yu\frac{du}{dy} - u^2 = 4y^2\ln(y) \iff \frac{u}{y}\frac{du}{dy} - \left(\frac{u}{y}\right)^2 = 4\ln(y). \tag{114}$$

The trick is to recognize the simplification $v = u/y$,

$$\frac{dv}{dy} = -\frac{u}{y^2} + \frac{1}{y}\frac{du}{dy} \iff vy\frac{dv}{dy} = \frac{u}{y}\frac{du}{dy} - \left(\frac{u}{y}\right)^2, \tag{115}$$

where I multiplied both sides by $u = vy$ in the second line. This simplifies things to a separable differential equation,

$$\begin{aligned} vy\frac{dv}{dy} = 4\ln(y) \iff \int v\,dv = 4\int \frac{\ln(y)}{y}\,dy \iff \frac{1}{2}v^2 &= 2\ln^2(y) + c_1 \\ u &= y\sqrt{4\ln^2(y) + c_1}, \end{aligned} \tag{116}$$

where I redefined arbitrary constants and recognized $v = u/y$. Since I started things off with $u = dy/dx$, this maps to

$$x + c_2 = \int \frac{dy}{y\sqrt{4\ln^2(y) + c_1}}, \tag{117}$$

which looks insane but simplifies with $\alpha = \ln(y)$:

$$\begin{aligned} x + c_2 = \int \frac{d\alpha}{\sqrt{4\alpha^2 + c_1}} \implies 2x + c_2 = \int \frac{d\alpha}{\sqrt{\alpha^2 + c_1^2}} &= \sinh^{-1}\left(\frac{\alpha}{c_1}\right) \\ &= \sinh^{-1}\left(\frac{\ln(y)}{c_1}\right), \end{aligned} \tag{118}$$

where I again freely redefined arbitrary constants. Inverting this equation, it's clear that

$$\begin{aligned} y(x) = e^{c_1 \sinh(c_2 + 2x)} &= \exp\left(c_1\left(\frac{e^{c_2+2x} + e^{-c_2-2x}}{2}\right)\right) \\ &= e^{c_1\left(c_2 e^{2x} + \frac{1}{c^2} e^{-2x}\right)}, \end{aligned} \tag{119}$$

or after continuing my journey of renaming constants,

$$y(x) = e^{c_1 e^{2x} + c_2 e^{-2x}} \tag{120}$$

is the most general solution of the ODE.

3 Series

3.1 Series 1

$$S=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\zeta(2n)}{\pi^{2n}}=\frac{1}{e^2-1} \tag{121}$$

Solution: Let's rewrite things and using the summation definition for $\zeta(2n)$,

$$\begin{aligned} S&=-\sum_{n=1}^{\infty}\left(-\frac{1}{\pi^2}\right)^n\zeta(2n)\\ &=-\sum_{n=1}^{\infty}\left(-\frac{1}{\pi^2}\right)^n\sum_{m=1}^{\infty}\frac{1}{m^{2n}}\\ &=-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(-\frac{1}{\pi^2m^2}\right)^n\\ &=-\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(-\frac{1}{\pi^2m^2}\right)\left(-\frac{1}{\pi^2m^2}\right)^n. \end{aligned} \tag{122}$$

Since the absolute value of the quantity in parentheses is always less than one for all m, I can evaluate the sum over n since it is geometric:

$$\begin{aligned} S&=-\sum_{m=1}^{\infty}\frac{-\frac{1}{\pi^2m^2}}{1+\frac{1}{\pi^2m^2}}\\ &=\frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2+\frac{1}{\pi^2}}. \end{aligned} \tag{123}$$

To evaluate this I will use the series expansion for coth,

$$\begin{aligned} \sum_{m=-\infty}^{\infty}\frac{1}{m^2+a^2}&=\frac{\pi}{a}\coth(\pi a)\\ \frac{1}{a^2}+2\sum_{m=1}^{\infty}\frac{1}{m^2+a^2}&=\frac{\pi}{a}\coth(\pi a)\iff\sum_{m=1}^{\infty}\frac{1}{m^2+a^2}=\frac{\pi}{2a}\coth(\pi a)-\frac{1}{2a^2}, \end{aligned} \tag{124}$$

such that S really is

$$S=\frac{1}{\pi^2}\left(\frac{\pi^2}{2}\coth(1)-\frac{\pi^2}{2}\right)=\frac{1}{2}\left(\frac{e+e^{-1}}{e-e^{-1}}-1\right)=\frac{1}{2}\left(\frac{2e^{-1}}{e-e^{-1}}\right), \tag{125}$$

$$\therefore\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\zeta(2n)}{\pi^{2n}}=\frac{1}{e^2-1} \tag{126}$$

3.2 Series 2

Show the following,

$$\sum_{m=-\infty}^{\infty} e^{-\frac{\alpha}{2}(\theta-2\pi m)^2} = \frac{1}{\sqrt{2\pi\alpha}} \sum_{n=-\infty}^{\infty} e^{in\theta-\frac{n^2}{2\alpha}}, \tag{127}$$

which is used in the Villain approximation of the 2D XY model in condensed matter physics.

Solution: I will perform a F.T.

$$S \equiv \sum_{m=-\infty}^{\infty} e^{-\frac{\alpha}{2}(\theta-2\pi m)^2} = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}, \tag{128}$$

where on the grounds of orthogonality

$$\begin{aligned} b_n &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-in\theta} \sum_{m=-\infty}^{\infty} e^{-\frac{\alpha}{2}(\theta-2\pi m)^2} \\ &= \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-in\theta} e^{-\frac{\alpha}{2}(\theta-2\pi m)^2}. \end{aligned} \tag{129}$$

The trick is to make the substitution $\phi = \theta - 2\pi m$, such that

$$\begin{aligned} b_n &= \sum_{m=-\infty}^{\infty} e^{2\pi imn} \int_{-\pi-2\pi m}^{\pi-2\pi m} \frac{d\phi}{2\pi} e^{-in\phi} e^{-\frac{\alpha}{2}\phi^2} \\ &= \sum_{m=-\infty}^{\infty} \int_{-\pi-2\pi m}^{\pi-2\pi m} \frac{d\phi}{2\pi} e^{-in\phi-\frac{\alpha}{2}\phi^2}, \end{aligned} \tag{130}$$

where I used the fact that $nm \in \mathbb{Z}$. The summation over m is now very simple,

$$b_n = \left[\cdots + \int_{\pi}^{3\pi} + \int_{-\pi}^{\pi} + \int_{-3\pi}^{-\pi} + \cdots \right] \frac{d\phi}{2\pi} e^{-in\phi-\frac{\alpha}{2}\phi^2} = \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} e^{-in\phi-\frac{\alpha}{2}\phi^2}, \tag{131}$$

and the remaining integral can be evaluated by completing the square,

$$\begin{aligned} b_n = e^{-\frac{n^2}{2\alpha}} \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} e^{-\frac{\alpha}{2}\phi^2 - in\phi + \frac{n^2}{2\alpha}} &= e^{-\frac{n^2}{2\alpha}} \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} e^{-\frac{\alpha}{2}(\phi-\frac{in}{\alpha})^2} \\ &= e^{-\frac{n^2}{2\alpha}} \frac{1}{2\pi} \sqrt{\frac{2\pi}{\alpha}}, \end{aligned} \tag{132}$$

and so (128) can be expressed as

$$\sum_{m=-\infty}^{\infty} e^{-\frac{\alpha}{2}(\theta-2\pi m)^2} = \frac{1}{\sqrt{2\pi\alpha}} \sum_{n=-\infty}^{\infty} e^{in\theta-\frac{n^2}{2\alpha}} \tag{133}$$

as desired.

3.3 Series 3

$$S = \sum_{k=1}^{\infty} \frac{\zeta(1-2k)}{\Gamma(2k-1)} = \frac{e}{(e-1)^2} - 1 \tag{134}$$

Solution: I immediately thought of using the relation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \tag{135}$$

and after playing around with it for a bit, I found that it works to use the reflection formula for the gamma function:

$$\Gamma(1-s) = \frac{\pi}{\sin(\pi s)\Gamma(s)} \implies \zeta(s) = 2^s \pi^s \frac{\sin\left(\frac{\pi s}{2}\right)}{\sin(\pi s)} \frac{\zeta(1-s)}{\Gamma(s)}. \tag{136}$$

Letting $s = 2k$ and moving things around,

$$\frac{\zeta(2k)}{2^{2k}\pi^{2k}} \frac{\sin(2\pi k)}{\sin(\pi k)} = \frac{\zeta(1-2k)}{\Gamma(2k)} = \frac{\zeta(1-2k)}{(2k-1)\Gamma(2k-1)} \iff \frac{\zeta(1-2k)}{\Gamma(2k-1)} = \frac{(2k-1)\cos(\pi k)}{2^{2k-1}\pi^{2k}} \zeta(2k), \tag{137}$$

where I used the double-angle identity for the sine. Since $k \in \mathbb{Z}$, I can replace $\cos(\pi k) \to (-1)^k$ inside the summation:

$$S = \sum_{k=1}^{\infty} \frac{(2k-1)(-1)^k}{2^{2k-1}\pi^{2k}} \zeta(2k) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(2k-1)(-1)^k}{2^{2k-1}\pi^{2k} n^{2k}}. \tag{138}$$

Here's the trick. Note that

$$\frac{2k-1}{n^{2k}} = -\frac{\partial}{\partial n}\left(\frac{1}{n^{2k-1}}\right). \tag{139}$$

Ignoring the question of passing the derivative across the sum (probably allowed by linearity), this gives

$$\begin{aligned} S = -\sum_{n=1}^{\infty} \frac{\partial}{\partial n} \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k-1}\pi^{2k} n^{2k-1}} = -2\sum_{n=1}^{\infty} \frac{\partial}{\partial n}\left[n \sum_{k=1}^{\infty}\left(-\frac{1}{4\pi^2 n^2}\right)^k\right] &= -2\sum_{n=1}^{\infty} \frac{\partial}{\partial n}\left[n\left(\sum_{k=0}^{\infty}\left(-\frac{1}{4\pi^2 n^2}\right)^k - 1\right)\right] \\ &= 2\sum_{n=1}^{\infty} \frac{\partial}{\partial n}\left[\frac{n}{1+4\pi^2 n^2}\right], \end{aligned} \tag{140}$$

where I manifested the k-series as geometric, and collected terms. Differentiating, I find

$$S = 2\sum_{n=1}^{\infty} \frac{1}{1+4\pi^2 n^2} - 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2}{(1+4\pi^2 n^2)^2}. \tag{141}$$

Looking at a table of series, it appears our friend will be

$$\sum_{n=0}^{\infty} \frac{1}{a^2+n^2} = \frac{1 + a\pi\coth(a\pi)}{2a^2}, \tag{142}$$

which I have no plan on deriving, yet. Clearly,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{1+4\pi^2 n^2} = \frac{1}{4\pi^2}\left(-(2\pi)^2 + \sum_{n=0}^{\infty} \frac{1}{(1/2\pi)^2 + n^2}\right) &= \frac{1}{4\pi^2}\left(-(2\pi)^2 + \frac{(2\pi)^2}{2}\left(1 + \frac{1}{2}\coth\left(\frac{1}{2}\right)\right)\right) \\ &= \frac{1}{4}\left(\coth\left(\frac{1}{2}\right) - 2\right). \end{aligned} \tag{143}$$

Such that one of the remaining series can be simplified,

$$S = \frac{1}{2}\left(\coth\left(\frac{1}{2}\right) - 2\right) - 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2}{(1+4\pi^2 n^2)^2}. \tag{144}$$

Now for the other,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n^2}{(1+4\pi^2n^2)^2} &= \frac{1}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{(1/2\pi)^2+n^2}{((1/2\pi)^2+n^2)^2}-\frac{1}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{(1/2\pi)^2}{((1/2\pi)^2+n^2)^2}\\ &= \frac{1}{(2\pi)^4}\left(-(2\pi^2)+\sum_{n=0}^{\infty}\frac{1}{(1/2\pi)^2+n^2}\right)-\frac{1}{(2\pi)^6}\sum_{n=1}^{\infty}\frac{1}{((1/2\pi)^2+n^2)^2}\\ &= \frac{1}{4(2\pi)^2}\left(\coth\left(\frac{1}{2}\right)-2\right)-\frac{1}{(2\pi)^6}\sum_{n=1}^{\infty}\frac{1}{((1/2\pi)^2+n^2)^2}\end{aligned} \tag{145}$$

where I added a zero in the form of $0=(1/2\pi)^2-(1/2\pi)^2$. Note that by differentiating (S5.9) we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{-2a}{(a^2+n^2)^2} &= -\frac{1}{a^3}-\frac{\pi}{2}\left(\frac{\coth(\pi a)}{a^2}+\frac{\pi\operatorname{csch}^2(\pi a)}{a}\right)\\ \iff \sum_{n=0}^{\infty}\frac{1}{(a^2+n^2)^2} &= \frac{1}{2a^4}+\frac{\pi}{4a}\left(\frac{\coth(\pi a)}{a^2}+\frac{\pi\operatorname{csch}^2(\pi a)}{a}\right),\end{aligned} \tag{146}$$

and so

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{((1/2\pi)^2+n^2)^2} = -(2\pi)^4+\sum_{n=0}^{\infty}\frac{1}{((1/2\pi)^2+n^2)^2} &= -(2\pi)^4+\frac{(2\pi)^4}{2}+\frac{\pi^2}{2}\left((2\pi)^2\coth\left(\frac{1}{2}\right)+2\pi^2\operatorname{csch}^2\left(\frac{1}{2}\right)\right)\\ &= -\frac{(2\pi)^4}{2}+\frac{\pi^2(2\pi)^2}{2}\left(\coth\left(\frac{1}{2}\right)+\frac{1}{2}\operatorname{csch}^2\left(\frac{1}{2}\right)\right).\end{aligned} \tag{147}$$

As a result,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n^2}{(1+4\pi^2n^2)^2} &= \frac{1}{4(2\pi)^2}\left(\coth\left(\frac{1}{2}\right)-2\right)+\frac{1}{2(2\pi)^2}-\frac{\pi^2}{2(2\pi)^4}\left(\coth\left(\frac{1}{2}\right)+\frac{1}{2}\operatorname{csch}^2\left(\frac{1}{2}\right)\right)\\ &= \frac{1}{32\pi^2}\left(\coth\left(\frac{1}{2}\right)-\frac{1}{2}\operatorname{csch}^2\left(\frac{1}{2}\right)\right).\end{aligned} \tag{148}$$

Plugging this into (144), we get

$$S=\frac{1}{2}\left(\coth\left(\frac{1}{2}\right)-2\right)-\frac{1}{2}\left(\coth\left(\frac{1}{2}\right)-\frac{1}{2}\operatorname{csch}^2\left(\frac{1}{2}\right)\right)=\frac{1}{4}\operatorname{csch}^2\left(\frac{1}{2}\right)-1. \tag{149}$$

Since $\operatorname{csch}^2(1/2)=4e/(e-1)^2$, this is simply

$$\sum_{k=1}^{\infty}\frac{\zeta(1-2k)}{\Gamma(2k-1)}=\frac{e}{(e-1)^2}-1 \tag{150}$$

which is quite nice.

4 Products

4.1 Product 1

$$A = \prod_{n=2}^{\infty} e\left(1-\frac{1}{n^2}\right)^{n^2} = \frac{\pi}{e\sqrt{e}} \tag{151}$$

Solution: The first major trick, which I think is the only way, is to take the natural logarithm of both sides,

$$\begin{aligned}\ln(A) = \ln\left(\prod_{n=2}^{\infty} e\left(1-\frac{1}{n^2}\right)^{n^2}\right) &= \sum_{n=2}^{\infty}\ln\left(e\left(1-\frac{1}{n^2}\right)^{n^2}\right)\\ &= \sum_{n=2}^{\infty} 1 + n^2\ln\left(1-\frac{1}{n^2}\right),\end{aligned} \tag{152}$$

giving a more manageable infinite sum. Next, I will represent the logarithm as an integral,

$$\ln\left(1-\frac{1}{n^2}\right) = -\int_0^1 \frac{du}{n^2-u}, \tag{153}$$

such that (152) becomes

$$\ln(A) = \sum_{n=2}^{\infty} 1 - n^2\int_0^1 \frac{du}{n^2-u}. \tag{154}$$

Now, since $u \in [0,1] \subset \mathbb{R}$ and $n \in [2,\infty) \subset \mathbb{N}$, we have that

$$\int_0^1 du = 1 \quad \text{and} \quad \frac{1}{n^2-u} > 0, \tag{155}$$

implying that the integral in (154) can be taken outside of everything, including the summation:

$$\ln(A) = \int_0^1 \left(\sum_{n=2}^{\infty} 1 - \frac{n^2}{n^2-u}\right) du = -\int_0^1 u\left(\sum_{n=2}^{\infty}\frac{1}{n^2-u}\right) du. \tag{156}$$

The next important trick is to use the following formula,

$$-\frac{\pi}{\sqrt{u}}\cot(\pi\sqrt{u}) = \sum_{n=-\infty}^{\infty}\frac{1}{n^2-u} = 2\left(\sum_{n=2}^{\infty}\frac{1}{n^2-u}\right) - \frac{1}{u} + \frac{2}{1-u}, \tag{157}$$

to replace the summation in (156), giving

$$\begin{aligned}\ln(A) = -\int_0^1 u\left(-\frac{\pi}{\sqrt{u}}\cot(\pi\sqrt{u}) + \frac{1}{2u} - \frac{1}{1-u}\right) du &= \frac{1}{2}\int_0^1 \left(\pi\sqrt{u}\cot(\pi\sqrt{u}) - 1 + \frac{2u}{1-u}\right) du\\ &\xrightarrow{u=v^2} \int_0^1 \left(\pi v^2\cot(\pi v) + \frac{2v^3}{1-v^2} - v\right) dv.\end{aligned} \tag{158}$$

For the middle term I will perform polynomial long-division,

$$\frac{2v^3}{1-v^2} = -2v + \frac{2v}{1-v^2}, \tag{159}$$

such that (158) becomes

$$\ln(A) = \int_0^1 \left(\pi v^2\cot(\pi v) - 3v + \frac{2v}{1-v^2}\right) dv = -\frac{3}{2} + \pi\int_0^1 v^2\cot(\pi v)\, dv + \int_0^1 \frac{2v}{1-v^2}\, dv. \tag{160}$$

Here is another cool trick. Substitute $v \to 1-v$ in the middle integral:

$$\begin{aligned}\int_0^1 v^2 \cot(\pi v)\, dv &\xrightarrow{v=1-v} -\int_1^0 (1-v)^2 \cot(\pi - \pi v)\, dv \\ &= -\int_0^1 (1-2v+v^2)\cot(\pi v)\, dv \\ &= -\int_0^1 v^2 \cot(\pi v)\, dv + 2\int_0^1 v\cot(\pi v)\, dv - \cancel{\int_0^1 \cot(\pi v)\, dx}.\end{aligned} \tag{161}$$

Here, I can add the original integral to both sides of the equation, then divide by two to give an interesting result

$$\int_0^1 \pi v^2 \cot(\pi v) dv = \int_0^1 \pi v \cot(\pi v) dv, \tag{162}$$

such that (160) becomes

$$\begin{aligned}\ln(A) &= -\frac{3}{2} + \int_0^1 \pi v \cot(\pi v) dv + \int_0^1 \frac{2v}{1-v^2} dv \\ &= -\frac{3}{2} + \lim_{\lambda\to 1^-}\int_0^\lambda \pi v \cot(\pi v) dv + \lim_{\lambda\to 1^-}\int_0^\lambda \frac{2v}{1-v^2} dv,\end{aligned} \tag{163}$$

where I took limits to account for the discontinuities. It is now time to explicitly evaluate the integrals. First,

$$\begin{aligned}\lim_{\lambda\to 1^-}\int_0^\lambda \frac{2v}{1-v^2} dv &= -\lim_{\lambda\to 1^-} \ln(1-\lambda^2) \\ &= -\lim_{\lambda\to 1^-}\ln(1+\lambda) - \lim_{\lambda\to 1^-}\ln(1-\lambda),\end{aligned} \tag{164}$$

which I expanded for future convenience. Second, by integration by parts we have

$$\begin{aligned}\lim_{\lambda\to 1^-}\int_0^\lambda \pi v\cot(\pi v) dv &= \lim_{\lambda\to 1^-} v\ln(\sin(\pi v))\Big|_0^\lambda - \int_0^1 \ln(\sin(\pi v)) dv \\ &= \lim_{\lambda\to 1^-}\lambda\ln(\sin(\pi\lambda)) - \cancel{\lim_{\alpha\to 0}\alpha\ln(\sin(\pi\alpha))} - \int_0^1 \ln(\sin(\pi v)) dv \\ &= \lim_{\lambda\to 1^-}\lambda\ln(\sin(\pi\lambda)) - \int_0^1 \ln(\sin(\pi v)) dv,\end{aligned} \tag{165}$$

where I noted that the middle limit vanishes. I can now substitute (164) and (165) into (163) to give

$$\begin{aligned}\ln(A) &= -\frac{3}{2} + \lim_{\lambda\to 1^-}\Big(\lambda\ln(\sin(\pi\lambda)) - \ln(1+\lambda) - \ln(1-\lambda)\Big) - \int_0^1 \ln(\sin(\pi v)) dv \\ &= -\frac{3}{2} - \ln(2) + \lim_{\lambda\to 1^-}\left[\lambda\ln\left(\frac{\sin(\pi\lambda)}{1-\lambda}\right)\right] - \int_0^1 \ln(\sin(\pi v)) dv.\end{aligned} \tag{166}$$

using L'Hopital's rule, along with the fact that the natural logarithm is continuous along its domain, implies

$$\lim_{\lambda\to 1^-}\left[\lambda\ln\left(\frac{\sin(\pi\lambda)}{1-\lambda}\right)\right] = \ln\left(\lim_{\lambda\to 1^-}\left[\frac{\sin(\pi\lambda)}{1-\lambda}\right]\right) = \ln(\pi), \tag{167}$$

such that all that remains of our original problem is

$$\ln(A) = -\frac{3}{2} + \ln(\pi) - \ln(2) - \int_0^1 \ln(\sin(\pi v)) dv. \tag{168}$$

To solve the last integral, I will first let $v \to 2v$:

$$\begin{aligned}\int_0^1 \ln(\sin(\pi v)) dv &= 2\int_0^{1/2} \ln(\sin(2\pi v)) dv \\ &= 2\int_0^{1/2} \ln(2) dv + 2\int_0^{1/2} \ln(\sin(\pi v)) dv + 2\int_0^{1/2} \ln(\cos(\pi v)) dv,\end{aligned} \tag{169}$$

where I used the double angle identity for $\sin(2\pi v)$ in the second step. Now, recognize that

$$\int_0^{1/2} \ln(\cos(\pi v))dv = \int_{1/2}^{1} \ln(\sin(\pi v))dv, \tag{170}$$

which allows me to combine the two integrals, giving

$$\int_0^1 \ln(\sin(\pi v))dv = \ln(2) + 2\int_0^1 \ln(\sin(\pi v))dv \implies \int_0^1 \ln(\sin(\pi v))dv = -\ln(2), \tag{171}$$

such that (168) can be written as

$$\ln(A) = -\frac{3}{2} + \ln(\pi) - \ln(2) + \ln(2) = -\frac{3}{2} + \ln(\pi). \tag{172}$$

To get the final answer, just exponentiate:

$$A = \pi e^{-3/2} = \frac{\pi}{e\sqrt{e}} \implies \prod_{n=2}^{\infty} e\left(1 - \frac{1}{n^2}\right)^{n^2} = \frac{\pi}{e\sqrt{e}} \tag{173}$$

4.2 Product 2

$$P = \prod_{n=1}^{\infty} \frac{(5n-2)(5n-3)}{(5n-1)(5n-4)} = \phi, \tag{174}$$

Here $\phi = (1+\sqrt{5})/2$ is the golden ratio.

Solution: To solve this I will use the Euler product representation

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}} \iff \frac{z\Gamma(z)}{5} = \prod_{n=1}^{\infty} \frac{1}{n^{z-1}} \frac{(n+1)^z}{5n+5z}, \tag{175}$$

such that

$$\begin{aligned}\left(-\frac{\Gamma\left(-\frac{1}{5}\right)}{25}\right)\left(-\frac{25}{2\left(-\frac{2}{5}\right)}\right)\left(-\frac{25}{3\Gamma\left(-\frac{3}{5}\right)}\right)\left(-\frac{4\Gamma\left(-\frac{4}{5}\right)}{25}\right) &= \prod_{n=1}^{\infty} \frac{(5n-2)(5n-3)}{(5n-1)(5n-4)} \\ &\quad \times \cancel{\frac{(n+1)^{-1/5}}{n^{-6/5}} \frac{n^{-7/5}}{(n+1)^{-2/5}} \frac{n^{-8/5}}{(n+1)^{-3/5}} \frac{(n+1)^{-4/5}}{n^{-9/5}}} \\ &= \prod_{n=1}^{\infty} \frac{(5n-2)(5n-3)}{(5n-1)(5n-4)},\end{aligned} \tag{176}$$

which is quite elegant. Therefore,

$$\begin{aligned} P = \frac{2}{3}\frac{\Gamma\left(-\frac{1}{5}\right)\Gamma\left(-\frac{4}{5}\right)}{\Gamma\left(-\frac{2}{5}\right)\Gamma\left(-\frac{3}{5}\right)} = \frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)} &= \frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(1-\frac{1}{5}\right)}{\Gamma\left(\frac{2}{5}\right)\Gamma\left(1-\frac{2}{5}\right)} \\ &= \frac{\sin\left(\frac{2\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)} \\ &= 2\cos\left(\frac{\pi}{5}\right) \\ &= \frac{1+\sqrt{5}}{2}, \quad \square \end{aligned} \tag{177}$$

where the second equality uses $\Gamma(n+1) = n\Gamma(n)$ and the fourth uses the Euler reflection formula.

5 Limits

5.1 Limit 1

$$L = \lim_{n\to\infty} \frac{(3n-1)!n^2}{27^n (n!)^3} = \frac{1}{2\pi\sqrt{3}} \tag{178}$$

Solution: For large n we can replace the factorials with the Stirling approximation,

$$n! \to \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad (3n-1)! \to \sqrt{2\pi(3n-1)}\frac{(3n-1)^{3n-1}}{e^{3n-1}}, \tag{179}$$

giving

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{n^2}{27^n}\frac{(2\pi(3n-1))^{1/2}}{(2\pi n)^{3/2}}\frac{(3n-1)^{3n-1}}{e^{3n-1}}\frac{e^{3n}}{n^{3n}} \\
&= \frac{e}{2\pi}\lim_{n\to\infty}\frac{1}{3^{3n}}\frac{(3n-1)^{3n-1/2}}{n^{3n-1/2}} \\
&= \frac{e}{2\pi\sqrt{3}}\lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^{3n-1/2} \\
&= \frac{e}{2\pi\sqrt{3}}\frac{\lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^{3n}}{\lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^{1/2}} \\
&= \frac{e}{2\pi\sqrt{3}}\lim_{n\to\infty}\exp\left(3n\ln\left(1-\frac{1}{3n}\right)\right) \\
&= \frac{e}{2\pi\sqrt{3}}\exp\left(-3\lim_{n\to\infty}\frac{1}{3(1-\frac{1}{3n})}\right) \implies L = \frac{1}{2\pi\sqrt{3}}.
\end{aligned} \tag{180}$$

I dragged the limit across the exponential since it is continuous and used l'Hopital's rule.

5.2 Limit 2

$$L = \lim_{x\to 0}\left(\frac{x^\pi - \pi^x}{x^e - e^x}\right)^{\csc(x)} = \frac{\pi}{e} \tag{181}$$

Solution: Do the $x = e^{\ln(x)}$ trick,

$$L = \lim_{x\to 0}\exp\left(\ln\left(\left(\frac{x^\pi - \pi^x}{x^e - e^x}\right)^{\csc(x)}\right)\right) = \exp\left(\lim_{x\to 0}\csc(x)\ln\left(\frac{x^\pi - \pi^x}{x^e - e^x}\right)\right) \tag{182}$$

where I dragged the limit through the exponential since it is continuous over its domain. For small x, a^x dominates x^a, and hence I can take $x^a - a^x \sim -a^x$ under the limit:

$$\begin{aligned}
L = \exp\left(\lim_{x\to 0}\csc(x)\ln\left(\frac{\pi^x}{e^x}\right)\right) &= \exp\left(\ln\left(\frac{\pi}{e}\right)\lim_{x\to 0}x\csc(x)\right) \\
&= \exp\left(\ln\left(\frac{\pi}{e}\right)\right) \\
&= \frac{\pi}{e},
\end{aligned} \tag{183}$$

where I used L'Hopital's rule trivially on the last limit.

5.3 Limit 3

$$L = \lim_{n\to\infty} \frac{\sqrt{n}}{e^n} \exp\left(\int_0^\infty \left\lfloor \frac{n}{e^x} \right\rfloor dx\right) = \frac{1}{\sqrt{2\pi}} \tag{184}$$

Solution: Clearly the integral should simplify. Let's do a substitution of $u = n/e^x$ such that

$$\begin{aligned}
\int_0^\infty \left\lfloor \frac{n}{e^x} \right\rfloor dx &= \int_0^n \frac{\lfloor u \rfloor}{u}\, du \\
&= \int_0^1 \frac{0}{u}\, du + \int_1^2 \frac{1}{u}\, du + \int_2^3 \frac{2}{u}\, du + ... + \int_{n-1}^n \frac{n-1}{u}\, du \\
&= \sum_{j=1}^{n-1} \int_j^{j+1} \frac{j}{u}\, du \\
&= \sum_{j=1}^{n-1} j(\ln(j+1) - \ln(j)) \\
&= 1(\ln(2) - \ln(1)) + 2(\ln(3) - \ln(2)) + ... + (n-1)(\ln(n) - \ln(n-1)) \\
&= -\ln(2) - \ln(3) - ... - \ln(n-1) + (n-1)\ln(n) \\
&= -\ln((n-1)!) + \ln(n^{n-1}) \\
&= \ln\left(\frac{n^{n-1}}{(n-1)!}\right).
\end{aligned} \tag{185}$$

In the second equality I used the fact that when $\lfloor u \rfloor$ is integrated it picks out the lower bound, and I made use of a semi-telescoping series with some trivial logarithm properties. Clearly then,

$$L = \lim_{n\to\infty} \frac{\sqrt{n}}{e^n} \frac{n^{n-1}}{(n-1)!}, \tag{186}$$

which can be simplified by noting that for large n,

$$(n-1)! \to \sqrt{2\pi(n-1)} \left(\frac{n-1}{e}\right)^{n-1}, \tag{187}$$

which is Stirling's approximation. Thus,

$$\begin{aligned}
L &= \lim_{n\to\infty} \sqrt{\frac{n}{2\pi(n-1)}} \left(\frac{e^{n-1}}{e^n}\right) \left(\frac{n}{n-1}\right)^{n-1} \\
&= \frac{1}{e\sqrt{2\pi}} \lim_{n\to\infty} \left(\frac{n}{n-1}\right)^n \\
&= \frac{1}{e\sqrt{2\pi}} \exp\left(\lim_{n\to\infty} n \ln\left(\frac{n}{n-1}\right)\right) = \frac{1}{e\sqrt{2\pi}} \exp\left(\lim_{n\to\infty} \left(\frac{n}{n-1}\right)\right) = \frac{1}{\sqrt{2\pi}}
\end{aligned} \tag{188}$$

Where in the second equality is used the fact that the limit of a product is a product of the limits, and in the fourth equality I used l'Hopital's rule.

5.4 Limit 4

$$L = \lim_{x\to 1^-}\left(\frac{1}{x}-1\right)\sum_{n=1}^{\infty} x^n \ln\left(1+x^n\right) = 2\ln(2)-1 \tag{189}$$

Solution: A neat trick is to express the logarithm as an integral and to expand the integrand in a common Maclaurin series, after which the integral can be performed and l'Hopital's rule can be used:

$$\begin{aligned}
L = \lim_{x\to 1^-}\left(\frac{1}{x}-1\right)\sum_{n=1}^{\infty} x^n \ln\left(1+y\right)\Big|_0^{x^n} &= \lim_{x\to 1^-}\left(\frac{1}{x}-1\right)\int_0^{x^n} dy \sum_{n=1}^{\infty}\frac{x^n}{1+y} \\
&= \lim_{x\to 1^-}\left(\frac{1}{x}-1\right)\int_0^{x^n} dy \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m x^n y^m \\
&= \lim_{x\to 1^-}\left(\frac{1}{x}-1\right)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\left(x^{m+2}\right)^n \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\lim_{x\to 1^-}\frac{\left(\frac{1}{x}-1\right)}{1-x^{m+2}} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)(m+2)}\lim_{x\to 1^-}\frac{-1/x^2}{x^{m+1}} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)(m+2)} \\
&= 2\ln(2)-1,
\end{aligned} \tag{190}$$

$$\therefore \lim_{x\to 1^-}\left(\frac{1}{x}-1\right)\sum_{n=1}^{\infty} x^n \ln\left(1+x^n\right) = 2\ln(2)-1 \tag{191}$$

which is quite an elegant result.