Solved Integrals

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Integral - 1

$$I = \int_0^{\pi/4} x \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) dx = 1 - \frac{\sqrt{2}}{2} \tag{1.1}$$

Solution: Let's expand the product for clarity,

$$\prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) = \cos\left(\frac{x}{2^1}\right)\cos\left(\frac{x}{2^2}\right)\cos\left(\frac{x}{2^3}\right) \times \ldots \times \cos\left(\frac{x}{2^k}\right). \tag{1.2}$$

The trick here is to use the double-angle formula for sine,

$$\sin(2x) = 2\sin(x)\cos(x) \implies \cos(x) = \frac{\sin(2x)}{2\sin(x)}, \tag{1.3}$$

such that (1.2) can be written as

$$\begin{aligned}
\prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) &= \frac{\sin(x)}{2\sin(x/2)} \frac{\sin(x/2)}{2\sin(x/4)} \frac{\sin(x/4)}{2\sin(x/6)} \times \ldots \times \frac{\sin(2x/2^k)}{2\sin(x/2^k)} \\
&= \sin(x) \lim_{k\to\infty} \frac{2^{-k}}{\sin(x/2^k)} \\
&= \sin(x) \lim_{k\to\infty} \frac{-k2^{-k-1}}{-xk2^{-k-1}\cos(x/2^k)} \\
&= \frac{\sin(x)}{x}
\end{aligned} \tag{1.4}$$

So (1.1) is simply

$$\begin{aligned}
I &= \int_0^{\pi/4} \sin(x) dx \\
&= -(\cos(\pi/4) - \cos(0)) \\
&= 1 - \frac{\sqrt{2}}{2},
\end{aligned} \tag{1.5}$$

$$\therefore \int_0^{\pi/4} x \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) dx = 1 - \frac{\sqrt{2}}{2} \tag{1.6}$$

Integral - 2

$$I = \int_{-\infty}^{\infty} \sin(e^x)dx = \frac{\pi}{2} \tag{2.1}$$

Solution: Let's first do a substitution of $u = e^x$,

$$I = \int_0^{\infty} \frac{\sin(u)}{u} du. \tag{2.2}$$

To solve this I will evaluate the contour integral

$$\oint_C \frac{e^{iz}}{z} dz \tag{2.3}$$

over the contour C defined in the figure below.

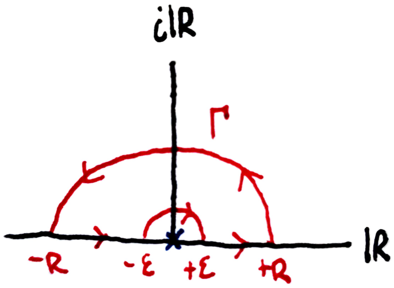

Figure 1: Problem (X-4)

To match (2.3) I will take the limits $\epsilon \to 0$ and $R \to \infty$. The smaller semicircle is a result of the fact that the integrand of (2.3) is singular at $z = 0$. The curve C is a concatenation of the four sub-curves, such that

$$\oint_C \frac{e^{iz}}{z} dz = \int_\Gamma \frac{e^{iz}}{z} dz + \int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_{-\epsilon}^{\epsilon} \frac{e^{iz}}{z} dz + \int_{\epsilon}^{R} \frac{e^{ix}}{x} dx = 0, \tag{2.4}$$

through Cauchy's residue theorem. Note that two of the integrals in (2.4) are purely real, since they lie on the real axis. Separating real from imaginary,

$$\int_\Gamma \frac{e^{iz}}{z} dz + \int_{-\epsilon}^{\epsilon} \frac{e^{iz}}{z} dz = -\int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx - \int_{\epsilon}^{R} \frac{e^{ix}}{x} dx = \int_{R}^{\epsilon} \frac{e^{ix} - e^{-ix}}{x} dx = -2i \int_{\epsilon}^{R} \frac{\sin(x)}{x} dx, \tag{2.5}$$

which is precisely the integral we want to solve. Let's rearrange and take our limits,

$$\begin{aligned} \lim_{(R,\epsilon)\to(\infty,0)} \int_{\epsilon}^{R} \frac{\sin(x)}{x} dx &= \frac{i}{2} \cancel{\lim_{(R,\epsilon)\to(\infty,0)} \int_\Gamma \frac{e^{iz}}{z} dz} + \frac{i}{2} \lim_{(R,\epsilon)\to(\infty,0)} \int_{-\epsilon}^{\epsilon} \frac{e^{iz}}{z} dz \\ &= \frac{i}{2} \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} \frac{e^{iz}}{z} dz, \end{aligned} \tag{2.6}$$

where I used Jordan's Lemma on the first integral.[1] The trick now is to parameterize the path by $z = \epsilon e^{i\phi}$ to give

$$\int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{1}{2} \lim_{\epsilon\to 0} \int_0^{\pi} e^{i\epsilon e^{i\phi}} d\phi = \frac{1}{2} \int_0^{\pi} \lim_{\epsilon\to 0} e^{i\epsilon e^{i\phi}} d\phi = \frac{1}{2} \int_0^{\pi} d\phi = \frac{\pi}{2}, \tag{2.7}$$

$$\therefore \int_{-\infty}^{\infty} \sin(e^x)dx = \frac{\pi}{2} \tag{2.8}$$

[1] If a complex function is defined on a complex semi-circular contour, then if the semi-circular contour is taken under the limit of arbitrarily large radius $R \to \infty$, the contour integral of the function on that contour goes to zero. However, certain constraints must be made.

Integral - 3

$$I = \int_0^{\pi} \ln(\sin(x))dx = -\pi \ln(2) \tag{3.1}$$

Solution: Complex analysis? Of course. Let's instead inspect the complex version

$$\begin{aligned} I &= \int_0^{\pi} \ln(\sin(z))dz \\ &= \int_0^{\pi} \ln\left(\frac{e^{iz} - e^{-iz}}{2i}\right) dz \\ &= \int_0^{\pi} \ln\left((e^{2iz} - 1)e^{-iz}\right) - \ln(2i)\, dz. \end{aligned} \tag{3.2}$$

Using the fact that

$$\ln(2i) = \ln(2) + \ln(i) = \ln(2) + \ln\left(e^{i\pi/2}\right) = \ln(2) + i\pi/2, \tag{3.3}$$

allows me to rewrite (3.2) as

$$I = \int_0^{\pi} \ln\left(e^{2iz} - 1\right) - iz - \ln(2) - i\pi/2\, dz. \tag{3.4}$$

In order to take advantage of Cauchy's residue theorem, we need to integrate over a contour. Inspecting (3.4) tells me that the bounds of the integral are indeed poles, as the natural log blows up there. So, I will choose the contour in the figure below,

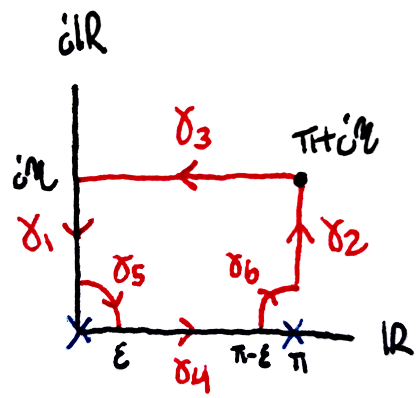

Figure 2: Problem (X-5)

I will take the limit $\eta \to \infty$ and $\epsilon \to 0$. Since all poles are avoided, CRT tells me that

$$I' = \oint_C = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} + \int_{\gamma_5} + \int_{\gamma_6} = 0, \tag{3.5}$$

where I am abusing notation. γ_4 is "real" nice,

$$\int_{\gamma_4} = \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi - \epsilon} \ln\left(e^{2ix} - 1\right) - ix - \ln(2) - i\pi/2\, dx = I, \tag{3.6}$$

and is precisely the integral I am trying to solve. If you are ever integrating over two parallel lines of the same length in the complex plane, it's best to do them together. So, for γ_1 I will parameterize the path by $z = it$ for $t \in [\epsilon, \eta]$, and for

γ_2 I will choose $z = \pi + it$ for the same t.

$$\int_{\gamma_1} + \int_{\gamma_2} = i \lim_{(\epsilon,\eta)\to(0,\infty)} \int_{\eta}^{\epsilon} \ln(e^{-2t} - 1) + t - \ln(2) - i\pi/2 \, dt$$
$$+ i \lim_{(\epsilon,\eta)\to(0,\infty)} \int_{\epsilon}^{\eta} \ln(e^{2i\pi - 2t} - 1) - i\pi + t - \ln(2) - i\pi/2 \, dt. \quad (3.7)$$

Since $e^{2i\pi} = 1$, when I flip the bounds of the second integral all terms cancel except for one:

$$\int_{\gamma_1} + \int_{\gamma_2} = i \lim_{(\epsilon,\eta)\to(0,\infty)} \int_{\eta}^{\epsilon} i\pi dt = \lim_{\eta\to\infty} \pi\eta. \quad (3.8)$$

this term is obviously problematic, so hopefully it gets canceled. I will parameterize γ_3 as $z = i\eta + \pi t$ for $t \in [1, 0]$,

$$\begin{aligned} \int_{\gamma_3} &= -\pi \lim_{\eta\to\infty} \int_0^1 \ln(e^{-2\eta + 2i\pi t} - 1) + \eta - i\pi t - \ln(2) - i\pi/2 \, dt \\ &= -\pi \int_0^1 \ln(-1) + \lim_{\eta\to\infty} \eta - i\pi t - \ln(2) - i\pi/2 \, dt \\ &= -\pi \left(\ln(-1) + \lim_{\eta\to\infty} \eta - i\pi/2 - \ln(2) - i\pi/2 \right) \end{aligned} \quad (3.9)$$

Since $\ln(-1) = i\pi$, the π terms cancel, leaving just

$$\int_{\gamma_3} = \pi \ln(2) - \lim_{\eta\to\infty} \pi\eta. \quad (3.10)$$

This is nice because when (3.10) is combined with (3.8) we get

$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} = \pi \ln(2), \quad (3.11)$$

and the diverging terms cancel out. So, (3.5) reads as

$$I = -\pi \ln(2) - \int_{\gamma_5} - \int_{\gamma_6}, \quad (3.12)$$

where I can parameterize γ_5 as $z = \epsilon e^{i\theta}$ for $\theta \in [\pi/2, 0]$, and γ_6 as $z = \pi + \epsilon e^{i\phi}$ for $\phi \in [\pi, \pi/2]$. The trick is that since both have the same form differential, $dz = i\epsilon e^{ix} dx$, when the limit that $\epsilon \to 0$ is taken the entire integral vanishes, and I can conclude that $I = -\pi \ln(2)$, which is purely real. Therefore,

$$\int_0^{\pi} \ln(\sin(x)) dx = -\pi \ln(2) \quad (3.13)$$

Integral - 4

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)^2} dx = \frac{\pi}{e} \tag{4.1}$$

Solution: Instead, let's look at the following,

$$\oint_C \frac{e^{iz}}{(z^2+1)^2} dz, \tag{4.2}$$

which is integrated over the contour below.

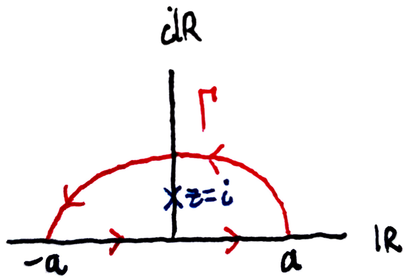

Figure 3: Problem (X-6)

I will eventually take the limit $a \to \infty$, and take the real part. Through Jordan's lemma, the semi-circular path portion tends to zero as $a \to \infty$, meaning the only contribution is the real part of (4.2), which is our original integral. Since the integrand has a pole at $z = i$, Cauchy's residue theorem tells me that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)^2} dx &= 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{(z^2+1)^2}, i\right) \\
&= 2\pi i \lim_{z \to i} \frac{d}{dz}\left(\frac{(z-i)^2 e^{iz}}{(z^2+1)^2}\right) \\
&= 2\pi i \lim_{z \to i} \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right) \\
&= 2\pi i \lim_{z \to i}\left(\frac{ie^{iz}(z+i)^2 - 2e^{iz}(z+i)}{(z+i)^4}\right) \\
&= 2\pi i \left(\frac{ie^{-1}(2i)^2 - 2e^{-1}(2i)}{(2i)^4}\right) \\
&= 2\pi i \left(-\frac{i}{2e}\right),
\end{aligned} \tag{4.3}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)^2} dx = \frac{\pi}{e} \tag{4.4}$$

Integral - 5

$$I = \int_0^1 \frac{x^2 - 1}{\ln(x)} dx = \ln(3) \tag{5.1}$$

Solution: As is standard for Feynman's trick, define a new integral

$$I(t) = \int_0^1 \frac{x^t - 1}{\ln(x)} dx, \tag{5.2}$$

for which I seek $I(2)$. The trick is to differentiate wrt the new variable t,

$$\begin{aligned} \frac{dI(t)}{dt} = \frac{d}{dt} \int_0^1 \frac{x^t - 1}{\ln(x)} dx = \int_0^1 \frac{\partial}{\partial t} \left(\frac{x^t - 1}{\ln(x)} \right) dx &= \int_0^1 \frac{1}{\ln(x)} \frac{\partial}{\partial t} \left(x^t \right) dx \\ &= \int_0^1 x^t dx \\ &= \frac{1^{t+1}}{t+1} \\ &= \frac{1}{t+1}. \end{aligned} \tag{5.3}$$

I can now integrate this to get what I want,

$$I(2) = \int \frac{1}{t+1} dt \Big|_{t=2} = \ln(t+1) \Big|_2 = \ln(3) \tag{5.4}$$

$$\therefore \int_0^1 \frac{x^2 - 1}{\ln(x)} dx = \ln(3) \tag{5.5}$$

Integral - 6

$$I = \int_0^{\infty} \ln\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4} \tag{6.1}$$

Solution: Use the logarithm's properties with a subsequent I.B.P,

$$\begin{aligned} I &= \int_0^{\infty} \ln\left(e^x+1\right) dx - \int_0^{\infty} \ln\left(e^x+1\right) dx \\ &= \left[x\ln(e^x+1) - x\ln(e^x-1)\right]\Big|_0^{\infty} - \int_0^{\infty} \frac{x}{1+e^{-x}}\, dx + \int_0^{\infty} \frac{x}{1-e^{-x}}\, dx. \end{aligned} \tag{6.2}$$

Let's inspect the boundary term,

$$\begin{aligned} \lim_{x\to\infty} x\ln\left(\frac{e^x+1}{e^x-1}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{e^x+1}{e^x-1}\right)}{1/x} &= \lim_{x\to\infty} \frac{\left(\frac{e^x-1}{e^x+1}\right)\left(\frac{e^x(e^x-1)-e^x(e^x+1)}{(e^x-1)^2}\right)}{-1/x^2} \\ &= 2\lim_{x\to\infty} \frac{x^2 e^x}{(e^x+1)(e^x-1)} \\ &= 2\lim_{x\to\infty} \frac{x^2}{e^x - e^{-x}} \\ &= 4\lim_{x\to\infty} \frac{x}{e^x + e^{-x}} \\ &= 0, \end{aligned} \tag{6.3}$$

$$\begin{aligned} \lim_{x\to 0} x\ln(e^x+1) - x\ln(e^x-1) = -\lim_{x\to 0} \frac{\ln(e^x-1)}{1/x} &= -\lim_{x\to 0} \frac{x^2}{1-e^{-x}} \\ &= -2\lim_{x\to 0} \frac{x}{e^{-x}} \\ &= 0, \end{aligned} \tag{6.4}$$

so the entire boundary term vanishes. Note that $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to 0} e^{-x} = 1$, so the integrands in (6.2) satisfy the condition to be represented as a geometric series,

$$\begin{aligned} I &= -\int_0^{\infty} \frac{x}{1+e^{-x}}\, dx + \int_0^{\infty} \frac{x}{1-e^{-x}}\, dx \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} \int_0^{\infty} x e^{-nx}\, dx + \sum_{n=0}^{\infty} \int_0^{\infty} x e^{-nx}\, dx, \end{aligned} \tag{6.5}$$

where I commuted the integral and sum because I am a physicist. Doing a little I.B.P,

$$\begin{aligned} I &= \left(\sum_{n=0}^{\infty} (-1)^{n+1} + \sum_{n=0}^{\infty}\right)\left(\cancel{-\frac{x}{n}e^{-nx}\Big|_0^{\infty}} + \frac{1}{n}\int_0^{\infty} e^{-nx}\, dx\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^2} + \sum_{n=0}^{\infty} \frac{1}{n^2} \\ &= \eta(2) + \zeta(2) = \frac{\pi^2}{12} + \frac{\pi^2}{6} \implies \int_0^{\infty} \ln\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4} \end{aligned} \tag{6.6}$$

Integral - 7 (Sophomore's Dream)

$$\int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n} \tag{7.1}$$

This was derived by Johann Bernoulli in 1697.

Proof: Let's fix the nested exponent with some logarithms,

$$\begin{aligned}\int_0^1 x^{-x}\,dx &= \int_0^1 e^{\ln\left(x^{-x}\right)}\,dx \\ &= \int_0^1 e^{-x\ln(x)}\,dx.\end{aligned} \tag{7.2}$$

Now, I need to get an infinite sum from somewhere, so how about I substitute the Maclaurin series for e^x,

$$\begin{aligned}\int_0^1 x^{-x}\,dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-x\ln(x))^n}{n!}\,dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n \ln^n(x)\,dx,\end{aligned} \tag{7.3}$$

where as a physicist I assumed the power series converges uniformly. This looks like it can be manipulated to a gamma function, so let $u = \ln(x)$ such that

$$\int_0^1 x^{-x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^0 e^{(n+1)u} u^n\,du. \tag{7.4}$$

Making the further substitution $u = -v/(n+1)$ to get the exponential how I want it,

$$\begin{aligned}\int_0^1 x^{-x}\,dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\infty}^0 \frac{(-1)^n}{(n+1)^n} v^n e^{-v} \left(-\frac{dv}{n+1}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n!(n+1)^{n+1}} \int_0^{\infty} v^n e^{-v} dv \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} \implies \int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n} \qquad \square\end{aligned} \tag{7.5}$$

In the 2nd-to-last step I recognized the integral as $\Gamma(n+1) = n!$ and shifted the index in the last step.

Integral - 8

$$I = \int_0^{1/2} \Gamma(1-x)\Gamma(1+x)\,dx = \frac{2}{\pi}G \approx 0.583 \tag{8.1}$$

where $G \approx 0.915$ is Catalan's constant.

Solution: The first few steps are to use the fact that $x\Gamma(x) = \Gamma(1+x)$, as well as the reflection property of the Gamma function:

$$\begin{aligned} I &= \int_0^{1/2} x\Gamma(x)\Gamma(1-x)\,dx \\ &= \int_0^{1/2} \frac{\pi x}{\sin(\pi x)}\,dx \\ &= \frac{1}{\pi}\int_0^{\pi/2} \frac{u}{\sin u}\,du. \end{aligned} \tag{8.2}$$

Maybe this will look better when I use the complex-exp version of sine,

$$I = \frac{2i}{\pi}\int_0^{\pi/2} \frac{u}{e^{iu}-e^{-iu}}\,du = \frac{2i}{\pi}\int_0^{\pi/2} \frac{u}{e^{iu}\left(1-e^{-2iu}\right)}\,du. \tag{8.3}$$

Since $e^{-2iu} \in [-1,1]$ on the bounds of integration $[0,\pi/2]$, it suffices to represent the term in the denominator in a geometric series,

$$I = \frac{2i}{\pi}\int_0^{\pi/2} \frac{u}{e^{iu}}\sum_{n=0}^{\infty} e^{-i2nu}\,du = \frac{2i}{\pi}\int_0^{\pi/2}\sum_{n=0}^{\infty} ue^{-iu(2n+1)}\,du \tag{8.4}$$

Does it makes sense to swap the integration and sum? Well, $e^{-iu(2n+1)}$ is bounded on the region of integration so I don't see why not:

$$\begin{aligned} I &= \frac{2i}{\pi}\sum_{n=0}^{\infty}\int_0^{\pi/2} ue^{-iu(2n+1)}\,du \\ &= \frac{2i}{\pi}\sum_{n=0}^{\infty}\left[-\frac{u}{i(2n+1)}e^{-iu(2n+1)}\Big|_0^{\pi/2} + \frac{1}{i(2n+1)}\int_0^{\pi/2} e^{-iu(2n+1)}\,du\right] \\ &= \frac{2i}{\pi}\sum_{n=0}^{\infty}\left[-\frac{\pi}{2i(2n+1)}e^{-i\pi(n+1/2)} + \frac{1}{(2n+1)^2}e^{-i\pi(n+1/2)} - \frac{1}{(2n+1)^2}\right] \end{aligned} \tag{8.5}$$

Now, it is simple to show that $e^{-i\pi(n+1/2)} = -i(-1)^n$, so

$$\begin{aligned} I &= \frac{2i}{\pi}\sum_{n=0}^{\infty}\left[\frac{\pi(-1)^n}{2(2n+1)} - \frac{i(-1)^n}{(2n+1)^2} - \frac{1}{(2n+1)^2}\right] \\ &= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} + i\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} - \frac{2i}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\ &= \frac{2}{\pi}G + i\left(\frac{\pi}{4}\right) - \frac{2i}{\pi}\left(\frac{\pi^2}{8}\right) \\ &= \frac{2}{\pi}G. \end{aligned} \tag{8.6}$$

$$\therefore \int_0^{1/2} \Gamma(1-x)\Gamma(1+x)\,dx = \frac{2}{\pi}G \approx 0.583 \tag{8.7}$$

Here, $G \approx 0.915$ is Catalan's constant. Quite nice how the two imaginary terms cancelled!

Integral - 9

$$\int_{-\infty}^{\infty} \frac{e^{-x^2}\sin\left(x^2\right)}{x^2}\,dx = \sqrt{\pi(2\sqrt{2}-2)} \tag{9.1}$$

Solution: Looks like a standard Feynman's trick to me,

$$\begin{aligned}
\Omega(a) = \int_{-\infty}^{\infty} \frac{e^{-ax^2}\sin\left(x^2\right)}{x^2}\,dx \implies \Omega'(a) &= -\int_{-\infty}^{\infty} e^{-ax^2}\sin\left(x^2\right)\,dx \\
&= -\frac{1}{2i}\int_{-\infty}^{\infty} e^{-ax^2}\left(e^{ix^2}-e^{-ix^2}\right)\,dx \\
&= -\frac{1}{2i}\int_{-\infty}^{\infty} e^{-(a-i)x^2}\,dx + \frac{1}{2i}\int_{-\infty}^{\infty} e^{-(a+i)x^2}\,dx \\
&= \frac{\sqrt{\pi}}{2i}\left(\frac{1}{\sqrt{a+i}}-\frac{1}{\sqrt{a-i}}\right),
\end{aligned} \tag{9.2}$$

where I used the standard result for a Gaussian integral we all know and love. Integrating backwards for $\Omega(a)$,

$$\begin{aligned}
\Omega(1) &= \frac{\sqrt{\pi}}{2i}\lim_{a\to 1}\int\left(\frac{1}{\sqrt{a+i}}-\frac{1}{\sqrt{a-i}}\right)\,da \\
&= \frac{\sqrt{\pi}}{i}\lim_{a\to 1}\left(\sqrt{a+i}-\sqrt{a-i}\right) \\
&= \frac{\sqrt{\pi}}{i}\left(\sqrt{1+i}-\sqrt{1-i}\right) \\
&= \frac{\sqrt{\pi}}{i}\left(2^{1/4}e^{i\pi/8}-2^{1/4}e^{-i\pi/8}\right) \\
&= 2\sqrt{\pi\sqrt{2}}\sin(\pi/8) \\
&= 2\sqrt{\pi\sqrt{2}}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right) \\
&= \sqrt{\pi(2\sqrt{2}-2)},
\end{aligned} \tag{9.3}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{-x^2}\sin\left(x^2\right)}{x^2}\,dx = \sqrt{\pi(2\sqrt{2}-2)} \tag{9.4}$$

Quite simple.

Integral - 10 (Vardi)

$$\int_{\pi/4}^{\pi/2} \ln(\ln(\tan x))\, dx = \frac{\pi}{2} \ln\left(\sqrt{2\pi} \cdot \frac{\Gamma(3/4)}{\Gamma(1/4)}\right) \tag{10.1}$$

This is known as the "Vardi" integral, first solved by Ilan Vardi in 1988.

Solution: The trick is to take the substitution $y = \ln(\tan x)$, from which results in a differential $\frac{dx}{dy} = e^y/(1+e^{2y})$:

$$\begin{aligned} I &= \int_0^\infty \frac{e^y}{1+e^{2y}} \ln(y)\, dy = \int_0^\infty \left(\frac{1}{1+e^{-2y}}\right) e^{-y} \ln(y)\, dy \\ &= \sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-(2n+1)y} \ln(y)\, dy. \end{aligned} \tag{10.2}$$

Making the substitution $u = (2n+1)y$,

$$\begin{aligned} I &= \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} \int_0^\infty e^{-u} \ln\left(\frac{u}{2n+1}\right) du = \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} \int_0^\infty e^{-u} \ln(u)\, du - \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} \ln(2n+1) \int_0^\infty e^{-u}\, du \\ &= -\gamma \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} - \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} \ln(2n+1), \end{aligned} \tag{10.3}$$

where I recognized the integral definition of the Euler-Mascheroni constant γ. The first sum can be simplified by recalling the series expansion of inverse tangent,

$$\tan^{-1}(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{2n+1}, \tag{10.4}$$

for which we have $\tan^{-1}(1) = \pi/4$. I will also replace the second summation by an equivalent representation,

$$I = -\frac{\gamma\pi}{4} - \sum_{n=1}^\infty \frac{\ln(n)}{n} \sin(n\pi/2), \tag{10.5}$$

where I used the fact that the terms in (10.3) run over odd integers and the sine factor takes those to zero anyway. To evaluate the remaining sum, I will take inspiration from the Fourier (Kummer) expansion of the log-gamma function,

$$\begin{aligned} \ln\Gamma(z) &= \left(\frac{1}{2} - z\right)(\gamma + \ln(2)) + (1-z)\ln(\pi) - \frac{1}{2}\ln(\sin(\pi z)) + \frac{1}{\pi}\sum_{n=1}^\infty \frac{\ln(n)}{n} \sin(2\pi n z) \\ \iff -\sum_{n=1}^\infty \frac{\ln(n)}{n} \sin\left(\frac{n\pi}{2}\right) &= \frac{\pi}{4}(\gamma + \ln(2)) + \frac{3\pi}{4}\ln(\pi) - \frac{\pi}{2}\ln\left(\frac{\sqrt{2}}{2}\right) - \pi \ln\Gamma\left(\frac{1}{4}\right), \end{aligned} \tag{10.6}$$

such that (10.5) can be written as

$$\begin{aligned} I &= -\frac{\gamma\pi}{4} + \frac{\pi}{4}(\gamma + \ln(2)) + \frac{3\pi}{4}\ln(\pi) - \frac{\pi}{2}\ln\left(\frac{\sqrt{2}}{2}\right) - \pi\ln\Gamma\left(\frac{1}{4}\right) \\ &= \frac{\pi}{2}\left(2\ln(\sqrt{2}) + \ln(\pi^{3/2}) + \ln\left(\Gamma^{-2}\left(\frac{1}{4}\right)\right)\right) \\ &= \frac{\pi}{2}\ln\left(\frac{2\pi^{3/2}}{\Gamma^2(1/4)}\right). \end{aligned} \tag{10.7}$$

Using the fact that $\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$, it follows that

$$\int_{\pi/4}^{\pi/2} \ln(\ln(\tan x))\, dx = \frac{\pi}{2} \ln\left(\sqrt{2\pi} \cdot \frac{\Gamma(3/4)}{\Gamma(1/4)}\right) \tag{10.8}$$

is the solution to the Vardi integral.

Integral - 11

$$I = \int_0^{\pi/2} \frac{\{\tan(x)\}}{\tan(x)}\,dx = \frac{1}{2}\left(\pi - \ln\left(\frac{\sinh(\pi)}{\pi}\right)\right) \tag{11.1}$$

Here, $\{\tan(x)\}$ is the fractional part of $\tan(x)$.

Solution: The fractional part satisfies $\{\tan(x)\} = \tan(x) - \lfloor \tan(x) \rfloor$, so

$$\begin{aligned} I &= \int_0^{\pi/2} 1 - \frac{\lfloor \tan(x) \rfloor}{\tan(x)}\,dx \\ &= \frac{\pi}{2} - \int_0^{\pi/2} \frac{\lfloor \tan(x) \rfloor}{\tan(x)}\,dx. \end{aligned} \tag{11.2}$$

Letting $u = \tan(x)$,

$$\begin{aligned} I &= \frac{\pi}{2} - \int_0^{\infty} \frac{\lfloor u \rfloor}{u(1+u^2)}\,du \\ &= \frac{\pi}{2} - \lim_{n\to\infty}\left[\int_0^1 \frac{0}{u(1+u^2)}\,du + \int_1^2 \frac{1}{u(1+u^2)}\,du + \cdots + \int_{n-1}^{n} \frac{n-1}{u(1+u^2)}\,du\right] \\ &= \frac{\pi}{2} - \sum_{j=1}^{\infty} j \int_j^{j+1} \frac{du}{u(1+u^2)} \\ &= \frac{\pi}{2} - \sum_{j=1}^{\infty} j \int_{\tan^{-1}(j)}^{\tan^{-1}(j+1)} \cot(x)\,dx. \end{aligned} \tag{11.3}$$

Since $\frac{d}{dx}\ln(\sin(x)) = \cot(x)$,

$$\begin{aligned} I &= \frac{\pi}{2} - \sum_{j=1}^{\infty} j\left(\ln\left(\sin\tan^{-1}(j+1)\right) - \ln\left(\sin\tan^{-1}(j)\right)\right) \\ &= \frac{\pi}{2} - \sum_{j=1}^{\infty} j\left(\ln\left(\frac{j+1}{\sqrt{1+(j+1)^2}}\right) - \ln\left(\frac{j}{\sqrt{1+j^2}}\right)\right) \\ &= \frac{\pi}{2} - \sum_{j=1}^{\infty}\left((j+1)\ln\left(\frac{j+1}{\sqrt{1+(j+1)^2}}\right) - j\ln\left(\frac{j}{\sqrt{1+j^2}}\right)\right) + \sum_{j=1}^{\infty}\ln\left(\frac{j+1}{\sqrt{1+(j+1)^2}}\right) \end{aligned} \tag{11.4}$$

Where I added a zero in order for the series to telescope properly. It is standard to show that the middle series has the partial sum

$$\begin{aligned} S_n &= -\ln\left(\frac{1}{\sqrt{2}}\right) + (n+1)\ln\left(\frac{n+1}{\sqrt{1+(n+1)^2}}\right) \\ \therefore \lim_{n\to\infty} S_n &= \frac{\ln(2)}{2}, \end{aligned} \tag{11.5}$$

such that (11.4) becomes

$$\begin{aligned} I &= \frac{\pi}{2} - \frac{\ln(2)}{2} + \sum_{j=1}^{\infty}\ln\left(\frac{j+1}{\sqrt{1+(j+1)^2}}\right) \\ &= \frac{\pi}{2} - \frac{\ln(2)}{2} - \frac{1}{2}\sum_{j=1}^{\infty}\ln\left(1 + \frac{1}{(j+1)^2}\right), \end{aligned} \tag{11.6}$$

where I did some manipulations in the second equality. The remaining series does not telescope, and I will rather look at the Euler product formula for sine,

$$\frac{\sin(x)}{x} = \prod_{j=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 j^2}\right) \implies \ln\left(\frac{\sin(x)}{x}\right) = \sum_{j=1}^{\infty}\ln\left(1 - \frac{x^2}{\pi^2 j^2}\right). \tag{11.7}$$

Substituting $x = i\pi$,

$$\begin{aligned} \ln\left(\frac{\sin(i\pi)}{i\pi}\right) &= \sum_{j=1}^{\infty}\ln\left(1 + \frac{1}{j^2}\right) \\ &= \ln(2) + \sum_{j=1}^{\infty}\ln\left(1 + \frac{1}{(j+1)^2}\right) \end{aligned} \tag{11.8}$$

Solving this equation for the series and substituting it into (11.6) results in the original integral having the form

$$I = \frac{\pi}{2} - \frac{\ln(2)}{2} + \frac{\ln(2)}{2} - \frac{1}{2}\ln\left(\frac{\sin(i\pi)}{i\pi}\right). \tag{11.9}$$

Since $\sin(i\pi) = i\sinh(\pi)$, we have the solution:

$$\int_0^{\pi/2} \frac{\{\tan(x)\}}{\tan(x)}\,dx = \frac{1}{2}\left(\pi - \ln\left(\frac{\sinh(\pi)}{\pi}\right)\right) \tag{11.10}$$

Integral - 12

$$I = \int_0^1 \int_0^1 \frac{\ln^s(x)}{1-xy}\, dxdy = (-1)^s \Gamma(s+1)\zeta(s+2) \tag{12.1}$$

Solution: The integral over y is really easy:

$$\begin{aligned} I &= \int_0^1 \ln^s(x) \left[\int_0^1 \frac{dy}{1-xy}\right] dx \\ &= -\int_0^1 \ln^s(x) \Big[\ln(1-xy)\Big]_0^1 \frac{dx}{x} \\ &= -\int_0^1 \ln^s(x)\ln(1-x)\ \frac{dx}{x}. \end{aligned} \tag{12.2}$$

The trick I'll employ is to use the series expansion for $\ln(1-x)$,

$$\begin{aligned} I &= -\int_0^1 \ln^s(x) \left(-\sum_{n=1}^{\infty} \frac{x^n}{n}\right) \frac{dx}{x} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n \ln^s(x) \frac{dx}{x}. \end{aligned} \tag{12.3}$$

Substituting $u = -n\ln(x)$,[2]

$$\begin{aligned} I &= (-1)^s \sum_{n=1}^{\infty} \frac{1}{n^{s+2}} \int_0^{\infty} u^s e^{-u} du \\ &= (-1)^s \Gamma(s+1) \sum_{n=1}^{\infty} \frac{1}{n^{s+2}} \\ &= (-1)^s \Gamma(s+1)\zeta(s+2), \end{aligned} \tag{12.4}$$

$$\therefore \int_0^1 \int_0^1 \frac{\ln^s(x)}{1-xy}\, dxdy = (-1)^s \Gamma(s+1)\zeta(s+2) \tag{12.5}$$

which is quite pretty.

[2] I did like three substitutions in a row on paper and I'm doing them all at once now that I have the hindsight.

Integral - 13

$$I = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \zeta(s)\Gamma(s) \tag{13.1}$$

Solution: Rewrite so that I can expand in a series:

$$\begin{aligned} I = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx &= \int_0^\infty \frac{x^{s-1}e^{-x}}{1 - e^{-x}}\, dx \\ &= \int_0^\infty x^{s-1}e^{-x}\sum_{n=0}^\infty e^{-nx}\, dx \\ &= \sum_{n=0}^\infty \int_0^\infty x^{s-1}e^{-(n+1)x}\, dx. \end{aligned} \tag{13.2}$$

Doing the substitution $u = (n+1)x$,

$$\begin{aligned} I = \sum_{n=0}^\infty \frac{1}{(n+1)^s}\int_0^\infty u^{s-1}e^{-u}\, du &= \Gamma(s)\sum_{n=1}^\infty \frac{1}{n^s} \\ &= \zeta(s)\Gamma(s) \implies \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \zeta(s)\Gamma(s) \end{aligned} \tag{13.3}$$

Pretty easy. Thie integrals shown up often in statistical mechanics, since the exponential term is reminiscent of the Bose-distribution.

Integral - 14

$$I = \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1}\, dx = \Gamma(n)\mathrm{Li}_n(z) \tag{14.1}$$

Solution: Let's do the same thing as (X-20):

$$\begin{aligned} I = z\int_0^\infty \frac{x^{n-1}e^{-x}}{1 - ze^{-x}}\, dx &= z\int_0^\infty x^{n-1}e^{-x}\sum_{m=0}^\infty z^m e^{-mx}\, dx \\ &= \sum_{m=0}^\infty z^{m+1}\int_0^\infty x^{n-1}e^{-(m+1)x}\, dx \\ &= \sum_{m=0}^\infty \frac{z^{m+1}}{(m+1)^n}\int_0^\infty u^{n-1}e^{-u}\, du \\ &= \Gamma(n)\sum_{m=1}^\infty \frac{z^m}{m^n} = \Gamma(n)\mathrm{Li}_n(z) \implies \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1}\, dx = \Gamma(n)\mathrm{Li}_n(z) \end{aligned} \tag{14.2}$$

Integral - 15

$$I = \int_{-\infty}^{\infty} e^{-\left(x^2+\frac{1}{x^2}\right)}\,dx = \frac{\sqrt{\pi}}{e^2} \tag{15.1}$$

Solution: Note that

$$x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2, \tag{15.2}$$

which allows me to write a nicer integral,

$$I = \frac{2}{e^2}\int_0^{\infty} e^{-(x-\frac{1}{x})^2}\,dx, \tag{15.3}$$

where I exploited the evenness. Make the substitution $x \to 1/x,\ dx \to -\frac{1}{x^2}dx$,

$$I = \frac{2}{e^2}\int_0^{\infty} e^{-(x-\frac{1}{x})^2}\frac{dx}{x^2}, \tag{15.4}$$

which is the same integral, so I can add the two and divide by two,

$$I = \frac{1}{e^2}\int_0^{\infty} e^{-(x-\frac{1}{x})^2}\left(1+\frac{1}{x^2}\right)dx. \tag{15.5}$$

Making the substitution $u = x - 1/x$, I find

$$I = \frac{1}{e^2}\int_{-\infty}^{\infty} e^{-u^2}\,du. \implies \int_{-\infty}^{\infty} e^{-\left(x^2+\frac{1}{x^2}\right)}\,dx = \frac{\sqrt{\pi}}{e^2} \tag{15.6}$$

which is a nice result.

Integral - 16

$$I = \int_0^{2\pi} d\phi \, \frac{1}{s^2 - 2sR\cos\phi + R^2} = \frac{2\pi}{|s^2 - R^2|} \tag{16.1}$$

This shows up commonly in many electrodynamics problems. Note that the denominator is the expansion of $||s\hat{r} - R\hat{r}'||^2$ in cylindrical coordinates in the case that $z = z' = \phi' = 0$.

Solution: Define the integral

$$I' = \int_0^{2\pi} d\phi \, \frac{1}{a + b\cos\phi}. \tag{16.2}$$

Clearly $|a| > |b|$ for finite-ness. Make the substitution that $z = e^{i\phi}$, such that $dz = iz \; d\phi$ and $2\cos\phi = z + z^{-1}$. This transforms the integration over $\phi \in \{0, 2\pi\}$ to an integration over the unit circle C in the complex plane:

$$\begin{aligned} I' = \oint \frac{dz}{iz} \frac{1}{a + \frac{b}{2}(z + z^{-1})} &= \frac{2}{ib} \oint dz \, \frac{1}{z^2 + \frac{2a}{b} z + 1} \\ &= \frac{2}{ib} \oint dz \, \frac{1}{(z - z_+)(z - z_-)}, \end{aligned} \tag{16.3}$$

where z_+, z_- are the poles of the integrand:

$$z_\pm = \frac{a}{b} \left[-1 \pm \sqrt{1 - \frac{b^2}{a^2}} \right] \tag{16.4}$$

Only z_+ is inside of the unit circle, and since it is a simple pole, Cauchy's Theorem tells me that the line integral (16.3) is $2\pi i$ times that residue,

$$\begin{aligned} I' = \frac{2}{ib}(2\pi i) \lim_{z \to z_+} \frac{(z - z_+)}{(z - z_+)(z - z_-)} &= \frac{4\pi}{b} \frac{1}{z_+ - z_-} \\ &= \frac{4\pi}{b} \frac{1}{\frac{2a}{b}\sqrt{1 - \frac{b^2}{a^2}}} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}}. \end{aligned} \tag{16.5}$$

This is already a neat result in itself. In the case that $a = s^2 + R^2$ and $b = -2sR$, I find

$$\int_0^{2\pi} d\phi \, \frac{1}{s^2 - 2sR\cos\phi + R^2} = \frac{2\pi}{\sqrt{s^4 - 2s^2R^2 + R^4}} = \frac{2\pi}{|s^2 - R^2|} \tag{16.6}$$

Integral - 17

Derive the following contour integral representation of the binomial coefficient,

$$I = \frac{1}{2\pi i}\oint_C dz\, \frac{(1+z)^y}{z^{x+1}} = \binom{y}{x}, \tag{17.1}$$

where C is the circle $|z| = 1$ traced CCW.

Solution: First let $z \to -z$,

$$I = \frac{1}{2\pi i}\frac{1}{(-1)^x}\oint_C dz\, \frac{(1-z)^y}{z^{x+1}} = \frac{e^{i\pi x}}{2\pi i}\oint_C dz\, \frac{(1-z)^y}{z^{x+1}}, \tag{17.2}$$

then parameterize C via $z = e^{i\theta}$, $d\theta = dz/iz$ such that

$$I = \frac{e^{i\pi x}}{2\pi}\int_0^{2\pi} d\theta\, \frac{\left(1-e^{i\theta}\right)^y}{e^{ix\theta}}. \tag{17.3}$$

The now trick is to use the binomial theorem,

$$(a+b)^c = \sum_{j=0}^{c}\binom{c}{j}a^{c-j}b^j \implies \left(1-e^{i\theta}\right)^y = \sum_{j=0}^{y}\binom{y}{j}(-1)^j e^{i\theta j}, \tag{17.4}$$

such that

$$\begin{aligned} I = \frac{e^{i\pi x}}{2\pi}\sum_{j=0}^{y}\binom{y}{j}(-1)^j\int_0^{2\pi} d\theta\, e^{i(j-x)\theta} &= \frac{e^{i\pi x}}{2\pi i}\sum_{j=0}^{y}\binom{y}{j}(-1)^j\frac{1}{j-x}\left(e^{2\pi i(j-x)}-1\right) \\ &= \frac{e^{-i\pi x}-e^{i\pi x}}{2\pi i}\sum_{j=0}^{y}\binom{y}{j}(-1)^j\frac{1}{j-x} \\ &= -\frac{\sin(\pi x)}{\pi}\sum_{j=0}^{y}\binom{y}{j}(-1)^j\frac{1}{j-x}, \end{aligned} \tag{17.5}$$

where I noted $e^{2\pi i(j-x)} = e^{-2\pi i x}$ since $j \in \mathbb{Z}$. Using the representation $(j-x)^{-1} = \int_0^1 dt\, t^{j-x-1}$,

$$\begin{aligned} I = -\frac{\sin(\pi x)}{\pi}\sum_{j=0}^{y}\binom{y}{j}(-1)^j\int_0^1 dt\, t^{j-x-1} &= -\frac{\sin(\pi x)}{\pi}\int_0^1 dt\, \left[\sum_{j=0}^{y}\binom{y}{j}(-t)^j\right]t^{-x-1} \\ &= -\frac{\sin(\pi x)}{\pi}\int_0^1 dt\, (1-t)^y t^{-x-1} \\ &= -\frac{\sin(\pi x)}{\pi}B(-x,1+y), \end{aligned} \tag{17.6}$$

where in the third equality I used the binomial theorem, after which I recognized a variant on the beta function. By the gamma function reflection formula and the properties of the beta function,

$$-\frac{\sin(\pi x)}{\pi} = \frac{1}{\Gamma(-x)\Gamma(1+x)}, \qquad B(-x,1+y) = \frac{\Gamma(-x)\Gamma(1+y)}{\Gamma(1+y-x)}, \tag{17.7}$$

I find that

$$\begin{aligned} I = \frac{1}{\Gamma(-x)\Gamma(1+x)}\frac{\Gamma(-x)\Gamma(1+y)}{\Gamma(1+y-x)} &= \frac{\Gamma(1+y)}{\Gamma(1+x)\Gamma(1+y-x)} \\ &= \frac{y!}{x!(y-x)!} \implies \frac{1}{2\pi i}\oint_C dz\, \frac{(1+z)^y}{z^{x+1}} = \binom{y}{x} \end{aligned} \tag{17.8}$$

as desired.

Integral - 18

$$I = \int_0^1 dx \, \left\{ \frac{1}{x} \right\} = 1 - \gamma \tag{18.1}$$

Here $\{x\}$ is the fractional part of x.

Solution: Let $y = 1/x$ such that $dx = -dy/y^2$ and

$$\begin{aligned} I = \int_1^\infty dy \, \frac{\{y\}}{y^2} = \int_0^\infty dy \, \frac{dy}{y} - \int_1^\infty dy \, \frac{\lfloor y \rfloor}{y^2} \\ &= \lim_{N\to\infty} \ln(N) - \left[\int_1^2 dy \, \frac{1}{y^2} + \int_2^3 dy \, \frac{2}{y^2} + \cdots \right] \\ &= \lim_{N\to\infty} \ln(N) - \lim_{N\to\infty} \sum_{n=1}^{N} \int_n^{n+1} dy \, \frac{n}{y^2} \\ &= \lim_{N\to\infty} \ln(N) - \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n+1} \\ &= 1 + \lim_{N\to\infty} \left[\ln(N) - \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} \right] = 1 - \gamma \implies \int_0^1 dx \, \left\{ \frac{1}{x} \right\} = 1 - \gamma \end{aligned} \tag{18.2}$$

where I noted $\{x\} = x - \lfloor x \rfloor$, and recognized the Euler-Mascheroni constant in its form as the limiting difference between the harmonic series and the natural logarithm.

Integral - 19

Evaluate the following integral,

$$I = \int_{-\pi/2}^{\pi/2} dx\, \frac{\cos(x)}{e^x + 1}, \tag{19.1}$$

by first proving the integral identity

$$\int_{-a}^{a} dx\, \frac{E(x)}{k(x)+1} = \frac{1}{2}\int_{-a}^{a} dx\, E(x), \tag{19.2}$$

where $E(x)$ is even for $x \in [-a, a]$ and $k(x)k(-x) = 1$.[3]

Solution: Let $x \to -x$ in (19.2):

$$\int_{-a}^{a} dx\, \frac{E(x)}{k(x)+1} = \int_{-a}^{a} dx\, \frac{E(x)}{k(-x)+1}. \tag{19.3}$$

Since these are equivalent, it follows that

$$\begin{aligned} \int_{-a}^{a} dx\, \frac{E(x)}{k(x)+1} &= \frac{1}{2}\int_{-a}^{a} dx\, \left[\frac{E(x)}{k(x)+1} + \frac{E(x)}{k(-x)+1}\right] \\ &= \frac{1}{2}\int_{-a}^{a} dx\, E(x)\left[\frac{k(x)+k(-x)+2}{k(x)k(-x)+k(x)+k(-x)+1}\right] \\ &= \frac{1}{2}\int_{-a}^{a} dx\, E(x) \quad \square \end{aligned} \tag{19.4}$$

and the identity is proved. For (19.1), I find

$$\int_{-\pi/2}^{\pi/2} dx\, \frac{\cos(x)}{e^x+1} = \frac{1}{2}\int_{-\pi/2}^{\pi/2} dx\, \cos(x) = 1, \tag{19.5}$$

without the need of any advanced techniques.

[3] I found this integral identity on instagram. The author didn't provide the proof, so I proved it and provided an additional example.

Integral - 20

$$\int_0^\infty dx\ \frac{e^{-ax}-e^{-bx}}{x\sec(px)} = \frac{1}{2}\ln\left(\frac{b^2+p^2}{a^2+p^2}\right) \tag{20.1}$$

Here $\mathrm{Re}(a), \mathrm{Re}(b) > |\mathrm{Im}(p)|$.

Solution: Let's go.

$$\begin{aligned}
I &= \int_0^\infty dx\ \frac{1}{x}\cos(px)e^{-ax} - \int_0^\infty dx\ \frac{1}{x}\cos(px)e^{-bx} \\
&= \frac{1}{2}\int_0^\infty dx\ \frac{1}{x}\left(e^{(ip-a)x}+e^{-(ip+a)x}\right) - \frac{1}{2}\int_0^\infty dx\ \frac{1}{x}\left(e^{(ip-b)x}+e^{-(ip+b)x}\right) \\
&= -\frac{1}{2}\int_0^a dy\int_0^\infty dx\ \left(e^{(ip-y)x}+e^{-(ip+y)x}\right) + \frac{1}{2}\int_0^b dy\int_0^\infty dx\ \left(e^{(ip-y)x}+e^{-(ip+y)x}\right) \\
&= -\frac{1}{2}\int_0^a dy\left(\frac{1}{y-ip}+\frac{1}{y+ip}\right) + \frac{1}{2}\int_0^b dy\left(\frac{1}{y-ip}+\frac{1}{y+ip}\right) \\
&= \frac{1}{2}\int_a^b dy\ \frac{2y}{y^2+p^2} \\
&= \frac{1}{2}\int_{a^2+p^2}^{b^2+p^2} du\ \frac{1}{u} \\
&= \frac{1}{2}\ln\left(\frac{b^2+p^2}{a^2+p^2}\right). \quad \square
\end{aligned} \tag{20.2}$$

Integral - 21

$$\iint_{[0,1]^2} dxdy\sqrt{\frac{x}{y}+\frac{y}{x}} = \frac{2\sqrt{2}}{3} + \frac{\Gamma^2(1/4)}{6\sqrt{\pi}} \tag{21.1}$$

Solution: Substitute $t = x/y$, $dx = ydt$, then perform an IBP over y using the FTC-1:

$$\begin{aligned} I = \int_0^1 dy\, y \int_0^{1/y} dt\, \sqrt{t+\frac{1}{t}} &= \frac{y^2}{2}\int_0^{1/y} dt\, \sqrt{t+\frac{1}{t}}\Big|_0^1 + \frac{1}{2}\int_0^1 dy\, \sqrt{y+\frac{1}{y}} \\ &= \frac{1}{2}\int_0^1 dt\, \sqrt{t+\frac{1}{t}} + \frac{1}{2}\int_0^1 dy\, \sqrt{y+\frac{1}{y}} \\ &= \int_0^1 dt\, \sqrt{t+\frac{1}{t}}, \end{aligned} \tag{21.2}$$

where the lower bound $y \to 0$ vanishes by noting that the integral will behave like $y^{-3/2}$, and so $y^2 y^{-3/2} = y^{1/2} \to 0$. The trick here is to make the substitution $y = u^2$ and perform another IBP,

$$\begin{aligned} I = 2\int_0^1 dt\, \sqrt{1+u^4} &= 2u\sqrt{1+u^4}\Big|_0^1 - 4\int_0^1 du\, \frac{u^4}{\sqrt{1+u^4}} \\ &= 2\sqrt{2} - 4\int_0^1 du\, \sqrt{1+u^4} + 4\int_0^1 du\, \frac{1}{\sqrt{1+u^4}}. \end{aligned} \tag{21.3}$$

I find the same integral on the r.h.s as the l.h.s, and hence

$$I = \frac{2\sqrt{2}}{3} + \frac{4}{3}\int_0^1 du\, \frac{1}{\sqrt{1+u^4}}. \tag{21.4}$$

Inspired by the beta function, substitute $u^2 = \tan(v)$,

$$\begin{aligned} I = \frac{2\sqrt{2}}{3} + \frac{2}{3}\int_0^{\pi/4} dv\, \frac{1}{\sqrt{\sin(v)\cos(v)}} &= \frac{2\sqrt{2}}{3} + \frac{2\sqrt{2}}{3}\int_0^{\pi/4} dv\, \frac{1}{\sqrt{\sin(2v)}} \\ &= \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{3}\int_0^{\pi/2} dw\, \sin^{-1/2}(w)\cos^0(w) \\ &= \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{6} B\left(1/4, 1/2\right) \\ &= \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{6}\frac{\Gamma^2(1/4)\Gamma(1/2)}{\Gamma(1/4)\Gamma(1/4+1/2)}. \end{aligned} \tag{21.5}$$

Using the Legendre duplication formula,

$$\Gamma(1/4)\Gamma(1/4+1/2) = 2^{1-2(1/4)}\sqrt{\pi}\Gamma(2(1/4)) = \sqrt{2\pi}\Gamma(1/2), \tag{21.6}$$

and so

$$\iint_{[0,1]^2} dxdy\sqrt{\frac{x}{y}+\frac{y}{x}} = \frac{2\sqrt{2}}{3} + \frac{\Gamma^2(1/4)}{6\sqrt{\pi}} \tag{21.7}$$

as desired.

Integral - 22

$$\int_0^\infty dx \int_0^\infty dy\; e^{-(x^n+y^n)} \ln\left[\frac{(xy)^{(xy)^n}}{(x^{x^n}y^{y^n})^{(xy)^n}}\right] = -\frac{2}{n^6}\Gamma^2\left(\frac{1}{n}\right)\left[2n+\psi\left(\frac{1}{n}\right)\right] \tag{22.1}$$

Solution: The log can be simplified as

$$\ln\left[\frac{(xy)^{(xy)^n}}{(x^{x^n}y^{y^n})^{(xy)^n}}\right] = x^ny^n\ln x + x^ny^n\ln y - x^{2n}y^n\ln x - x^ny^{2n}\ln y, \tag{22.2}$$

and so

$$\begin{aligned} I &= \int_0^\infty dx \int_0^\infty dy\; e^{-(x^n+y^n)}\left[x^ny^n\ln x + x^ny^n\ln y - x^{2n}y^n\ln x - x^ny^{2n}\ln y\right] \\ &= 2\int_0^\infty dx \int_0^\infty dy\; e^{-(x^n+y^n)}\left[x^ny^n\ln x - x^{2n}y^n\ln x\right], \end{aligned} \tag{22.3}$$

where I used the fact that the integrand is symmetric under $x \leftrightarrow y$. The y-integral is simple after the substitution $u = y^n$, in which

$$\begin{aligned} I &= \frac{2}{n}\int_0^\infty dx\; e^{-x^n}\left[x^n\ln x\int_0^\infty du\; u^{1/n}e^{-u} - x^{2n}\ln x\int_0^\infty du\; u^{1/n}e^{-u}\right] \\ &= \frac{2}{n}\Gamma\left(\frac{1}{n}+1\right)\int_0^\infty dx\left[\ln x\; x^ne^{-x^n} - \ln x\; x^{2n}e^{-x^n}\right]. \end{aligned} \tag{22.4}$$

Note that $\Gamma'(z) = \psi(z)\Gamma(z)$, where

$$\Gamma'(z) = \frac{d}{dz}\int_0^\infty dx\; x^{z-1}e^{-x} = \int_0^\infty dx\;\ln x\; x^{z-1}e^{-x}, \tag{22.5}$$

which motivates in (22.4) the substitution $u = x^n$, giving

$$\begin{aligned} I &= \frac{2}{n^3}\Gamma\left(\frac{1}{n}\right)\int_0^\infty du\left[\ln(u^{1/n})\; u^{1/n}e^{-u} - \ln(u^{1/n})\; u^{1/n+1}e^{-u}\right] \\ &= \frac{2}{n^4}\Gamma\left(\frac{1}{n}\right)\left[\Gamma'\left(\frac{1}{n}+1\right) - \Gamma'\left(\frac{1}{n}+2\right)\right] \\ &= \frac{2}{n^4}\Gamma\left(\frac{1}{n}\right)\left[\psi\left(\frac{1}{n}+1\right)\Gamma\left(\frac{1}{n}+1\right) - \psi\left(\frac{1}{n}+2\right)\Gamma\left(\frac{1}{n}+2\right)\right] \\ &= \frac{2}{n^5}\Gamma^2\left(\frac{1}{n}\right)\left[\psi\left(\frac{1}{n}+1\right) - \left(\frac{1}{n}+1\right)\psi\left(\frac{1}{n}+2\right)\right]. \end{aligned} \tag{22.6}$$

Using the fact that $\psi(z+1) = \psi(z) + \frac{1}{z}$, then

$$\begin{aligned} I &= \frac{2}{n^5}\Gamma^2\left(\frac{1}{n}\right)\left[\psi\left(\frac{1}{n}\right) + n - \left(\frac{1}{n}+1\right)\left(\psi\left(\frac{1}{n}\right) + n + \frac{1}{\frac{1}{n}+1}\right)\right] \\ &= \frac{2}{n^5}\Gamma^2\left(\frac{1}{n}\right)\left[-\frac{1}{n}\psi\left(\frac{1}{n}\right) - 2\right] \\ &= -\frac{2}{n^6}\Gamma^2\left(\frac{1}{n}\right)\left[2n+\psi\left(\frac{1}{n}\right)\right], \quad \square \end{aligned} \tag{22.7}$$

as desired.

Integral - 23

$$\int_0^\infty \frac{dx}{x^n+1} = \frac{\pi}{n\sin(\pi/n)} \tag{23.1}$$

Solution: Let $u = x^n$, such that $dx = \frac{1}{n}u^{1/n-1}du$ and

$$I_n = \frac{1}{n}\int_0^\infty du\, \frac{u^{1/n-1}}{1+u}. \tag{23.2}$$

Recall the Beta function

$$B(z_1, z_2) = \int_0^\infty dt\, \frac{t^{z_1-1}}{(1+t)^{z_1+z_2}} = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}. \tag{23.3}$$

Comparing, I find that $z_1 = 1/n$ and $z_1 + z_2 = 1$. Hence,

$$I_n = \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{n}-1\right)}{n\Gamma(1)} = \frac{\pi}{n\sin(\pi/n)} \tag{23.4}$$

where I made use of Euler's reflection formula

Integral - 24

$$I = \int_0^\infty dx\ \tan^{-1}(x^{-k}) = \frac{\pi}{2}\sec\left(\frac{\pi}{2k}\right) \tag{24.1}$$

Here $k > 1$. I found this guy on Youtube (Maths505), but the solution is my own.

Solution: Like any arctangent integral, IBP gives

$$I = x\tan^{-1}(x^{-k})\Big|_0^\infty + k\int_0^\infty \frac{dx\ x}{x^{k+1}(1+x^{-2k})} = k\int_0^\infty \frac{dx}{x^k(1+x^{-2k})}. \tag{24.2}$$

Letting $u = x^{1-k}$ results in

$$I = \frac{k}{k-1}\int_0^\infty \frac{du}{1+u^{2k/(k-1)}}, \tag{24.3}$$

where my goal now is to express this in terms of the Beta function via a second substitution $v = u^{2k/(k-1)}$, giving

$$\begin{aligned} I = \frac{1}{2}\int_0^\infty dv\ \frac{v^{-(k+1)/2k}}{1+v} &= \frac{1}{2}B\left(\frac{k-1}{2k}, 1-\frac{k-1}{2k}\right) \\ &= \frac{\pi}{2\sin(\pi(k+1)/2k)} \implies I = \frac{\pi}{2}\sec\left(\frac{\pi}{2k}\right), \quad k > 1 \end{aligned} \tag{24.4}$$

where I used Euler's reflection formula. The Maths505 channel gives an interesting case for $k = 5$, where since $\cos(\pi/5) = \phi/2$, where ϕ is the Golden ratio, we have

$$\int_0^\infty dx\ \tan^{-1}\left(\frac{1}{x^5}\right) = \frac{\pi}{\phi}, \tag{24.5}$$

which is pretty cool.