Solved Differential Equations

Ethan Huecker

Last Update: Mar 2026

Difeq - 1

$$\frac{dy}{dx}\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 = 2\left(\frac{d^2y}{dx^2}\right)^2 \implies y(x) = c_1 + c_2\tanh^{-1}(c_3e^x) \tag{1.1}$$

Solution: The lowest order derivative is 1st-order, so substitute $\frac{dy}{dx} = u$, such that

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = u\frac{du}{dy}, \qquad \frac{d^3y}{dx^3} = \frac{du}{dx}\frac{du}{dy} + u\frac{d^2u}{dxdy} = u\left(\frac{du}{dy}\right)^2 + u^2\frac{d^2u}{dy^2}. \tag{1.2}$$

are the transformations of the higher order derivatives. Thus, (1.1) becomes

$$\begin{aligned} u^2\left[\left(\frac{du}{dy}\right)^2 + u\frac{d^2u}{dy^2}\right] + u^2 &= 2u^2\left(\frac{du}{dy}\right)^2 \\ u\frac{d^2u}{dy^2} &= \left(\frac{du}{dy}\right)^2 - 1, \end{aligned} \tag{1.3}$$

and we successively dropped an order. Continuing with this process, let $\frac{du}{dy} = v$ such that $\frac{d^2u}{dy^2} = v\frac{dv}{du}$ and (1.3) becomes

$$\begin{aligned} uv\frac{dv}{du} &= v^2 - 1 \\ \iff \int \frac{v\ dv}{v^2-1} &= \int \frac{du}{u} \\ \frac{1}{2}\ln|v^2-1| &= \ln|u| + C \\ \ln\left|\frac{v^2-1}{u^2}\right| &= C \implies \frac{v^2-1}{u^2} = C, \end{aligned} \tag{1.4}$$

where I changed the arbitrary constant $C \in \mathbb{R}$ many times. Solving this equation for $v = \frac{du}{dy}$ gives

$$\begin{aligned} \frac{du}{dy} &= \sqrt{1+Cu^2} \\ \iff \int \frac{du}{\sqrt{1+Cu^2}} &= \int dy \\ \frac{1}{\sqrt{C}}\sinh^{-1}(\sqrt{C}u) &= y + D \\ u &= \frac{1}{C}\sinh(Cy+D). \end{aligned} \tag{1.5}$$

where I changed the constant $\sqrt{C} \to C$ and absorbed it into the second constant $D \in \mathbb{R}$. Since we know that $u = \frac{dy}{dx}$, we actually have

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{C}\sinh(Cy+D) \\
\iff \int \operatorname{csch}(Cy+D)\; dy &= \frac{1}{C}\int dx \\
\xrightarrow{z=Cy+D} \frac{1}{C}\int \operatorname{csch} z\; dz &= \frac{1}{C}x + E \\
\int \frac{dz}{e^z - e^{-z}} &= \frac{x}{2} + E.
\end{aligned} \tag{1.6}
$$

Where I absorbed arbitrary constants into one another and defined $z = Cy + D$ to simplify the remaining integral. Continuing, multiply the top and bottom of the integrand by e^z to make use of a u-sub,

$$
\begin{aligned}
\frac{x}{2} + E &= \int \frac{e^z\; dz}{e^{2z}-1} \\
&\xrightarrow{\zeta=e^z} \int \frac{d\zeta}{\zeta^2 - 1} \\
&= \frac{1}{2}\int \frac{d\zeta}{\zeta-1} - \frac{1}{2}\int \frac{d\zeta}{\zeta+1} \\
&= \frac{1}{2}\ln|\zeta-1| - \frac{1}{2}\ln|\zeta+1|\,.
\end{aligned} \tag{1.7}
$$

Making use of a trivial log property and multiplying by 2 (absorbing it into constants),

$$
\begin{aligned}
x + E &= \ln\left|\frac{\zeta-1}{\zeta+1}\right| \\
&= \ln\left|\frac{e^z-1}{e^z+1}\right| \\
&= \ln|\tanh(z)| \\
&= \ln|\tanh(Cy+D)|.
\end{aligned} \tag{1.8}
$$

Taking the exponential and the hyperbolic arc-tangent gives

$$
\begin{aligned}
\tanh(Cy+D) &= Ee^x \\
Cy + D &= \tanh(Ee^x) \implies y(x) = c_1 + c_2\tanh^{-1}(c_3e^x)
\end{aligned} \tag{1.9}
$$

which is the solution for arbitrary constants c_1, c_2, c_3.

Difeq - 2

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 4y^2\ln(y) \implies y(x) = e^{c_1 e^{2x} + c_2 e^{-2x}} \tag{2.1}$$

Solution: Make the substitution $u = \frac{dy}{dx}$, from which

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{dy}{dx}\frac{du}{dy} = u\frac{du}{dy}, \tag{2.2}$$

such that

$$yu\frac{du}{dy} - u^2 = 4y^2\ln(y) \iff \frac{u}{y}\frac{du}{dy} - \left(\frac{u}{y}\right)^2 = 4\ln(y). \tag{2.3}$$

The trick is to recognize the simplification $v = u/y$,

$$\frac{dv}{dy} = -\frac{u}{y^2} + \frac{1}{y}\frac{du}{dy} \iff vy\frac{dv}{dy} = \frac{u}{y}\frac{du}{dy} - \left(\frac{u}{y}\right)^2, \tag{2.4}$$

where I multiplied both sides by $u = vy$ in the second line. This simplifies things to a separable differential equation,

$$\begin{aligned} vy\frac{dv}{dy} = 4\ln(y) \iff \int v\,dv = 4\int \frac{\ln(y)}{y}\,dy \iff \frac{1}{2}v^2 &= 2\ln^2(y) + c_1 \\ u &= y\sqrt{4\ln^2(y) + c_1}, \end{aligned} \tag{2.5}$$

where I redefined arbitrary constants and recognized $v = u/y$. Since I started things off with $u = dy/dx$, this maps to

$$x + c_2 = \int \frac{dy}{y\sqrt{4\ln^2(y) + c_1}}, \tag{2.6}$$

which looks insane but simplifies with $\alpha = \ln(y)$:

$$\begin{aligned} x + c_2 = \int \frac{d\alpha}{\sqrt{4\alpha^2 + c_1}} \implies 2x + c_2 &= \int \frac{d\alpha}{\sqrt{\alpha^2 + c_1^2}} \\ &= \sinh^{-1}\left(\frac{\alpha}{c_1}\right) \\ &= \sinh^{-1}\left(\frac{\ln(y)}{c_1}\right), \end{aligned} \tag{2.7}$$

where I again freely redefined arbitrary constants. Inverting this equation, it's clear that

$$\begin{aligned} y(x) &= e^{c_1 \sinh(c_2 + 2x)} \\ &= \exp\left(c_1\left(\frac{e^{c_2+2x} + e^{-c_2-2x}}{2}\right)\right) \\ &= e^{c_1\left(c_2 e^{2x} + \frac{1}{c^2} e^{-2x}\right)}, \end{aligned} \tag{2.8}$$

or continuing my journey of renaming constants,

$$y(x) = e^{c_1 e^{2x} + c_2 e^{-2x}} \tag{2.9}$$

is the most general solution of the dif-eq (2.1).